A Kähler–Ricci flow proof of the Wu–Yau Theorem

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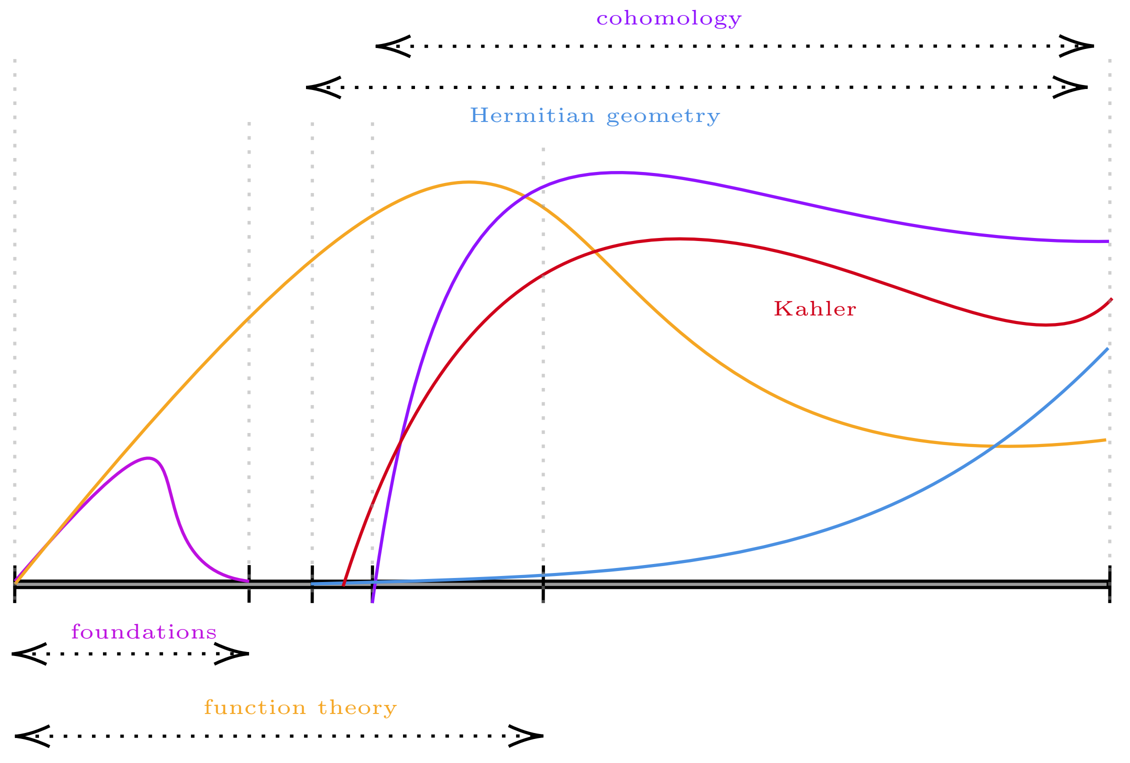
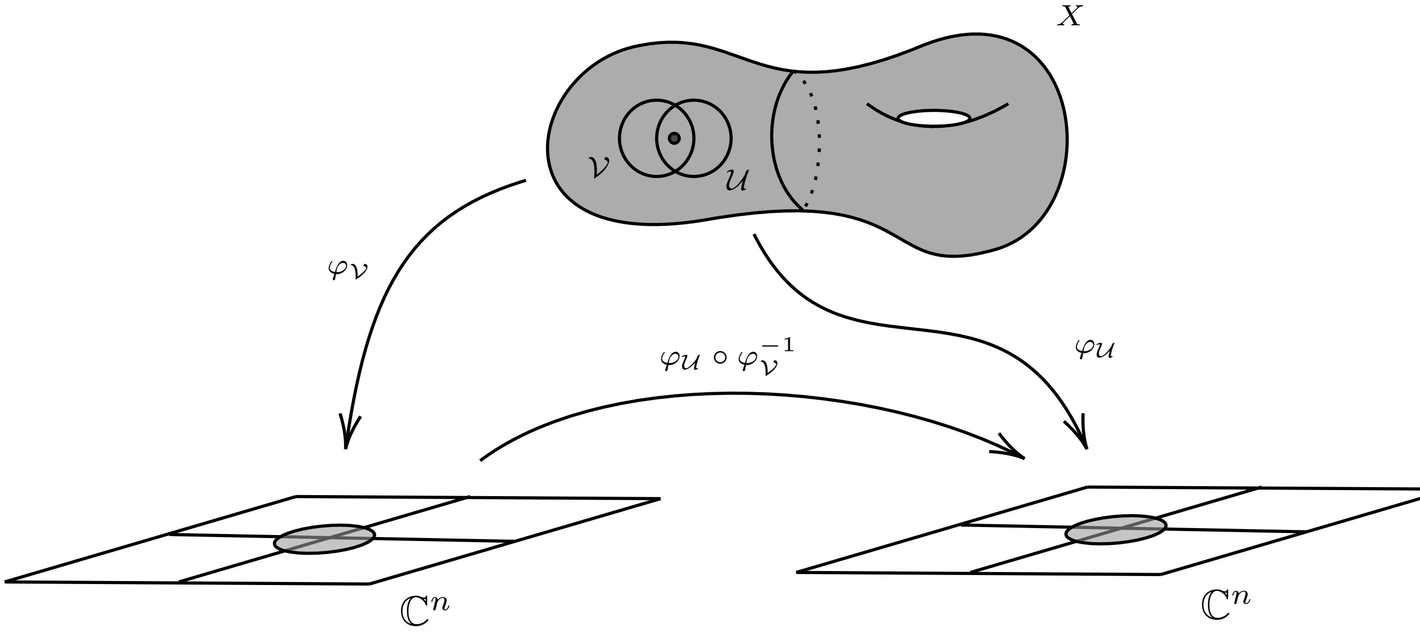
Lecture 1: Initiation and Propaganda

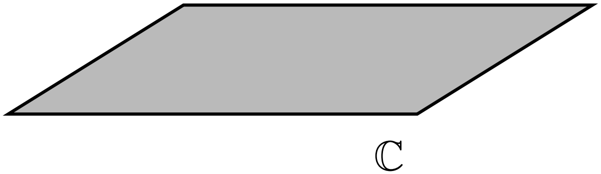

cohomology
Hermitian geometry
Kahler
foundations
function theory

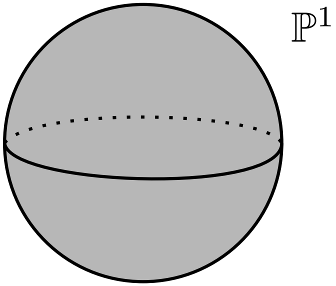
A complex manifold is a (connected, second-countable) Hausdorff topological space X with an atlas whose transition maps are holomorphic.



(†) Complex Euclidean space $\mathbb{C}^n$.

(†) Complex projective space $\mathbb{P}^n$ – the quotient of $(\mathbb{C}^{n+1})^\times$ by the multiplicative action of $\mathbb{C}^\times$.



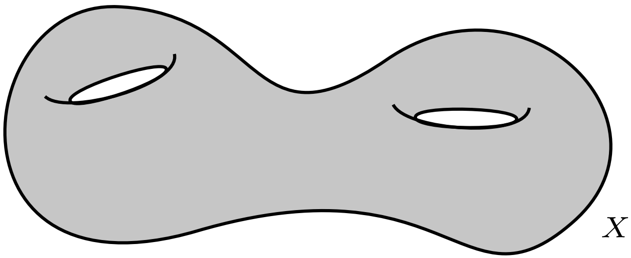


Complex manifolds are significantly more rigid than their real counterparts.

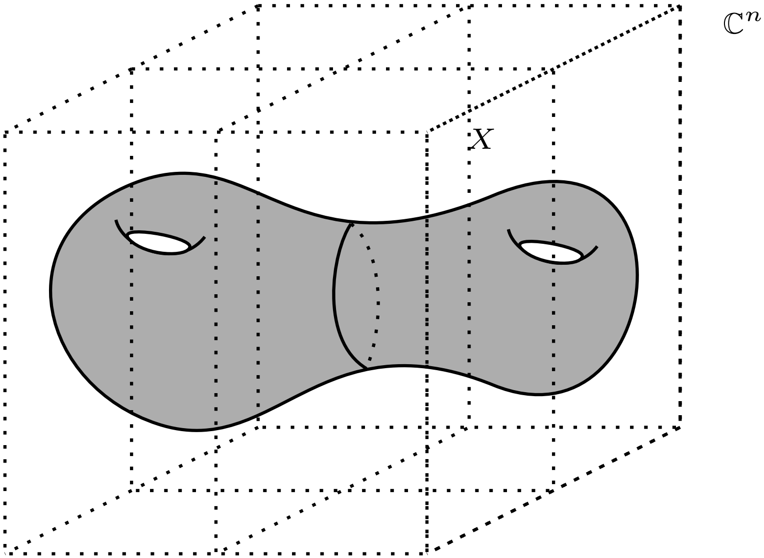
The most notable example is the failure of the holomorphic-analogue of the Whitney embedding theorem:

A smooth manifold M (smoothly) embeds into $\mathbb{R}^N$ for some N.

Suppose X is a compact complex manifold (e.g., $\mathbb{P}^n$ or a compact Riemann surface) which holomorphically embeds into some $\mathbb{C}^n$.



The coordinate functions on $\mathbb{C}^n$ restrict to X, yielding holomorphic functions on a compact set.



This violates the maximum principle unless X is a point.

If a compact complex manifold X (holomorphically) embeds into some $\mathbb{C}^n$, then X is a point.

Those complex manifolds which holomorphically embed into $\mathbb{C}^n$ form an important class of complex manifolds – Stein manifolds.

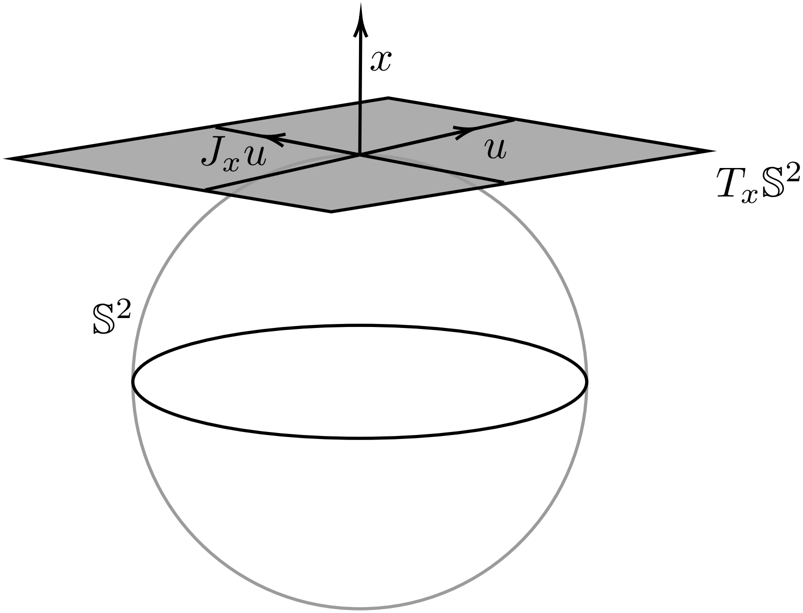
Stein manifolds generalize the notion of a domain of holomorphy.

Any open Riemann surface is Stein.

We want to understand how the presence of a complex structure interacts with the geometry of the manifold.

An almost complex structure is a smooth section

$$J \in H^0(M, \mathrm{End}(TM)), \qquad J^2 = -\mathrm{id}.$$



A complex manifold is easily shown to be an almost complex manifold.

Existence of an almost complex structure can be formulated in terms of the existence of a section of a vector bundle, characteristic classes give obstructions to finding an almost complex structure.

The only almost complex spheres are $\mathbb{S}^2$ and $\mathbb{S}^6$.

Almost complex manifolds are not complex, in general.

The obstruction is measured by the Nijenhuis tensor

$$N^J(u,v) \;:=\; [u,v] + J[Ju,v] + J[u,Jv] - [Ju,Jv].$$

The Newlander–Nirenberg theorem states that $N^J \equiv 0$ is equivalent to the existence of the existence of local holomorphic coordinates.

An almost complex structure J is said to be a complex structure if

$$N^J \equiv 0.$$

Every almost complex structure on a Riemann surface is a complex structure.

Let V be a complex vector space with a positive-definite Hermitian form

$$H : V \times V \to \mathbb{C}.$$

We may write

$$H(u, v) \;=\; g(u, v) - \sqrt{-1}\omega(u, v),$$

where $g := \mathrm{Re}(H)$ and $\omega := -\mathrm{Im}(H)$.

(i) g defines a positive-definite quadratic form on V,

(ii) ω defines a non-degenerate $(1, 1)$–form[1] on V.

[1] A form ω is of type $(1, 1)$ if $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$.

Let $\mathcal{H}$ denote the space of Hermitian forms on TX. Write

$$\mathcal{H} \; = \; \mathcal{R} \oplus \mathcal{S},$$

where $\mathcal{R} = \{\text{quadratic forms}\}$, and $\mathcal{S} = \{(1,1)\text{–forms}\}$.

The almost complex structure J defines a linear embedding

$$\Phi_J : \mathcal{R} \longrightarrow \Lambda^{1,1}(T^*X), \qquad\qquad \Phi_J(g) \; = \; \omega(\cdot,\cdot) := g(J\cdot,\cdot).$$

The linear constraint

$$d\omega \; = \; 0$$

on the bundle $\mathcal{R}$ defines a Kähler structure.

The Kähler condition can be equivalently formulated as the almost complex structure being Levi-Civita–parallel

$$\nabla^{\mathrm{LC}} J = 0.$$

Kähler manifolds are precisely those complex manifolds which support compatible symplectic and Riemannian structures.

If X is compact, the Kähler condition $d\omega = 0$ implies ω represents a non-trivial cohomology class in $H^2_{\mathrm{DR}}(X, \mathbb{R})$.

A cohomology class $\alpha \in H^2_{\mathrm{DR}}(X, \mathbb{R})$ is said to be a Kähler class if α is represented by a Kähler metric.

Complex Euclidean space $\mathbb{C}^n$ with its Euclidean metric

$$\omega_{\mathbb{C}^n} \; := \; \sqrt{-1} \sum_{i,j=1}^{n} dz_i \wedge d\overline{z}_j$$

is Kähler.

Complex submanifolds of Kähler manifolds are Kähler.

In particular, all Stein manifolds are Kähler.

Many complex manifolds support non-Kähler structures or no Kähler structures at all.

Hopf surfaces $\mathbb{S}^3 \times \mathbb{S}^1$ cannot support a Kähler structure since

$$b_2(\mathbb{S}^3 \times \mathbb{S}^1) = 0.$$

A (partial) example:

Suppose $\mathbb{S}^6$ supports a complex structure.

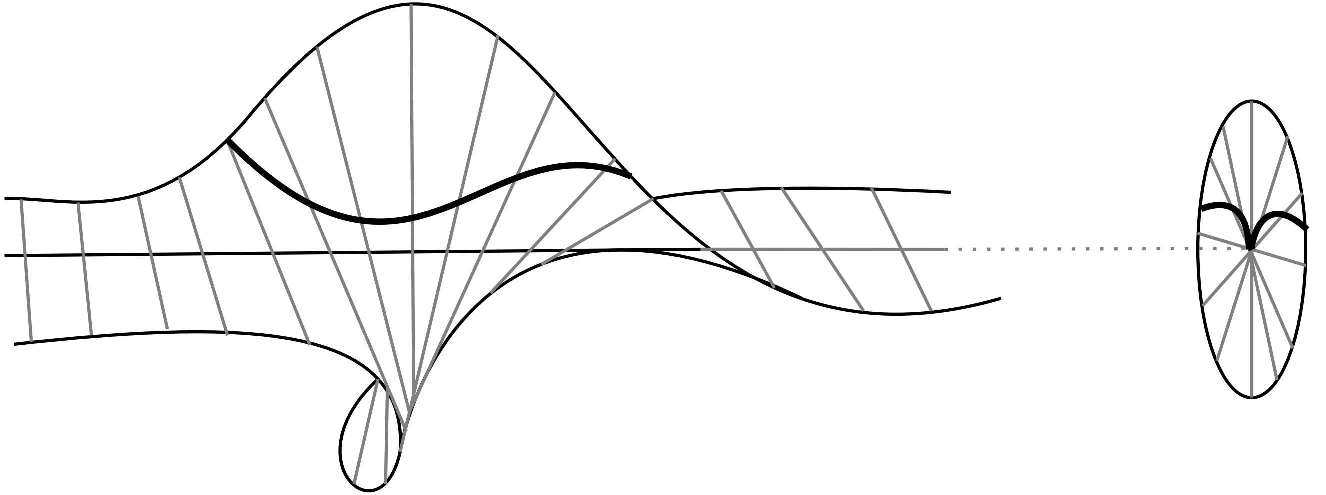
Since $b_2(\mathbb{S}^6) = 0$, there is no Kähler structure on $\mathbb{S}^6$.

Campana–Demailly–Peternell showed that $\mathbb{S}^6$ does not support any non-constant meromorphic functions (i.e., the algebraic dimension $a(\mathbb{S}^6) = 0$).

The blow up

$$\mathrm{Bl}_p(\mathbb{S}^6) \to \mathbb{S}^6$$

of $\mathbb{S}^6$ at one point is diffeomorphic to $\mathbb{P}^3$.



But $\mathrm{Bl}_p(\mathbb{S}^6) \simeq \mathbb{P}^3$ is not biholomorphic to the standard $\mathbb{P}^3$, since

$$a(\mathrm{Bl}_p(\mathbb{S}^6)) = 0.$$

So $\mathbb{S}^6$ parametrizes a family of exotic complex (non-Kähler) structures on $\mathbb{P}^3$.

Complex projective space $\mathbb{P}^n$ with its Fubini–Study metric

$$\omega_{\mathrm{FS}} \;:=\; \sqrt{-1}\partial\bar{\partial}\log\left(1+\sum_{k=1}^{n}|z_k|^2\right)$$

is Kähler.

Projective manifolds are Kähler.

Not all Kähler manifolds are projective:

Standard example: Sufficiently generic complex torus

$$\mathbb{C}^{n>1}/\Lambda.$$

Let X be a complex manifold.

A (holomorphic) line bundle $\mathcal{L} \to X$ is said to be ample if the sections of a sufficiently high tensor power $\mathcal{L}^{\otimes k}$ furnish a holomorphic embedding

$$\Phi : X \longrightarrow \mathbb{P}^{N_k}.$$

For instance, the tangent bundle $T\Sigma$ to a Riemann surface Σ of genus $g \geq 2$ is ample.

Let $\mathcal{L} \to X$ be a line bundle. A Hermitian metric h on $\mathcal{L}$ is given by a smooth family of Hermitian metrics

$$h_p : \mathcal{L}_p \times \mathcal{L}_p \to \mathbb{C}$$

on the fibers $\mathcal{L}_p$ of $\mathcal{L}$.

If

$$K_X := \Lambda^{n,0}(T^*X)$$

denotes the canonical bundle. A Hermitian metric on K_X is given by a volume form.

The curvature form of a Hermitian metric is given by

$$\Theta_h = \sqrt{-1}\partial\overline{\partial}\log(h).$$

This defines a closed real $(1,1)$–form.

A line bundle $\mathcal{L} \to X$ is positive if there is a smooth Hermitian metric h such that

$$\Theta_h > 0$$

(in the sense of positive forms).

If $\mathcal{L}^{-1} \to X$ is the dual bundle (with the induced metric h^{-1}), then the curvature form

$$\Theta_{h^{-1}} = \sqrt{-1}\partial\overline{\partial}\log(h^{-1}) = -\sqrt{-1}\partial\overline{\partial}\log(h) = -\Theta_h.$$

A line bundle is negative if its dual bundle is positive.

Since the curvature form Θ_h is closed, it represents a cohomology class

$$[\Theta_h] \in H^2_{\mathrm{DR}}(X, \mathbb{R}).$$

This cohomology class is the first Chern class of $\mathcal{L}$, denoted $c_1(\mathcal{L})$.

Define the curvature tensor R of a Kähler metric:

$$R_{i\ \ k\bar{\ell}}^{\ m} \;:=\; -\partial_{\bar{\ell}}\Gamma^m_{ik}.$$

If $A = (A_{i\bar{j}})$ is an invertible Hermitian matrix with entries depending on t. Then Cramer's rule gives

$$\frac{d}{dt}\det(A) \;=\; A^{i\bar{j}}\left(\frac{d}{dt}A_{i\bar{j}}\right)\det(A).$$

Hence,

$$\mathrm{Ric}_{i\bar{j}} \;=\; -\partial_{\bar{j}}\Gamma^k_{ki} \;=\; -\partial_{\bar{j}}(g^{k\bar{q}}\partial_i g_{k\bar{q}}) \;=\; -\partial_{\bar{j}}\partial_i \log(\det(g)).$$

The Ricci curvature of a Kähler metric is locally given by

$$\mathrm{Ric}_\omega \;=_{\mathrm{loc}}\; -\sqrt{-1}\partial\overline{\partial}\log(\omega^n).$$

A Hermitian metric h on the canonical bundle K_X is equivalent to a volume form ω^n.

The curvature form is

$$\Theta_h = \sqrt{-1}\partial\overline{\partial}\log(\omega^n).$$

The Ricci curvature is the curvature form of a Hermitian metric on the anti-canonical bundle K_X^{-1}:

$$\mathrm{Ric}_\omega = -\sqrt{-1}\partial\overline{\partial}\log(\omega^n) = \Theta_{h^{-1}}.$$

The Ricci curvature Ric_ω of a Kähler metric ω is cohomological in nature, representing the first Chern class of the anti-canonical bundle

$$c_1(K_X^{-1}) = [\mathrm{Ric}_\omega].$$

Negative Ricci implies K_X is a positive line bundle:

$$\mathrm{Ric}_\omega < 0 \implies {K_X}^{-1} < 0 \implies K_X > 0.$$

If (X, ω) is compact Kähler with $\mathrm{Ric}_\omega < 0$, then K_X is ample.

This follows from the famous Kodaira embedding theorem:

> A positive line bundle over a compact Kähler manifold is ample.

The Ricci flow starting from a Kähler metric ω_0 is given by a family of Riemannian metrics g_t such that

$$\frac{\partial g_t}{\partial t} = -\mathrm{Ric}_{g_t}, \qquad g|_{t=0} = g_0.$$

The Ricci flow preserves the Kähler condition, and the resulting flow is called the Kähler–Ricci flow.

A cohomology class in $H^2_{\mathrm{DR}}(X, \mathbb{R})$ is called a Kähler class if it is represented by a Kähler form.

The set of Kähler classes in the $H^2_{\mathrm{DR}}(X, \mathbb{R})$ form an open convex cone – the Kähler cone.

A cohomology class $[\alpha] \in H^2_{\mathrm{DR}}(X, \mathbb{R})$ on the boundary of the Kähler cone is called a nef class.

Taking cohomology classes of the Kähler–Ricci flow:

$$\frac{\partial}{\partial t}[\omega_t] \;=\; -[\mathrm{Ric}_{\omega_t}] \;=\; 2\pi c_1(K_X).$$

Hence,

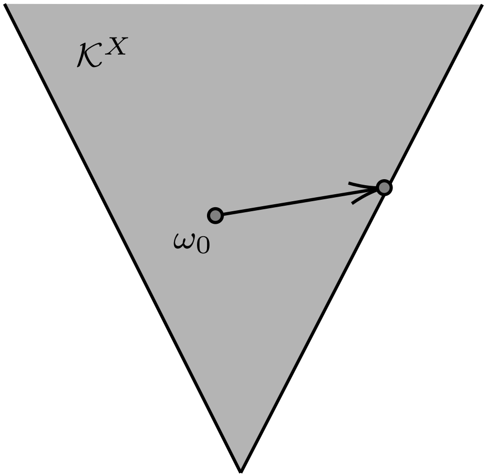
$$[\omega_t] \;=\; [\omega_0] + 2\pi t c_1(K_X).$$

Let (X^n, ω_0) be a compact Kähler manifold.

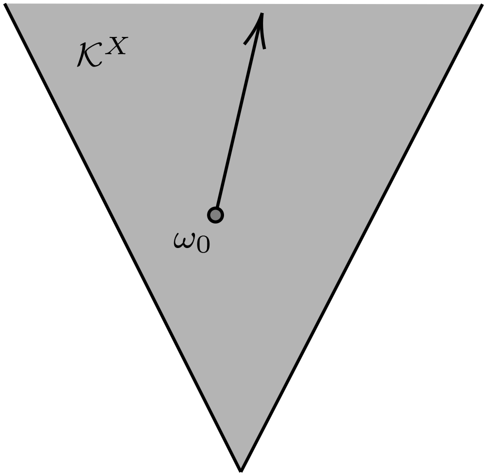
Then the Kähler–Ricci flow has a unique solution ω_t defined on the maximal time interval $[0, T)$, where

$$T \;:=\; \sup\{t > 0 : [\omega_0] + 2\pi t c_1(K_X) \text{ is Kähler}\}.$$

The Kähler–Ricci flow exists for all time $\iff$ the canonical bundle K_X is nef.



$T < +\infty$



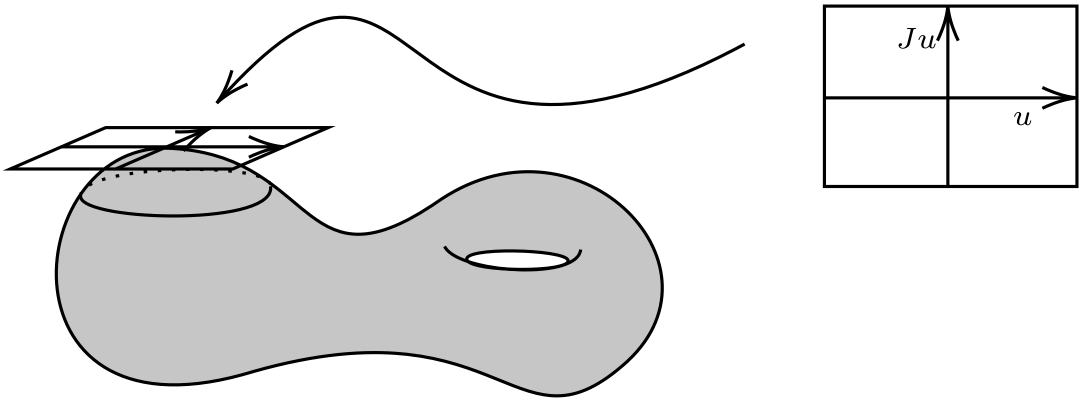
$T = +\infty$

The sectional curvature of a metric ω defines a function

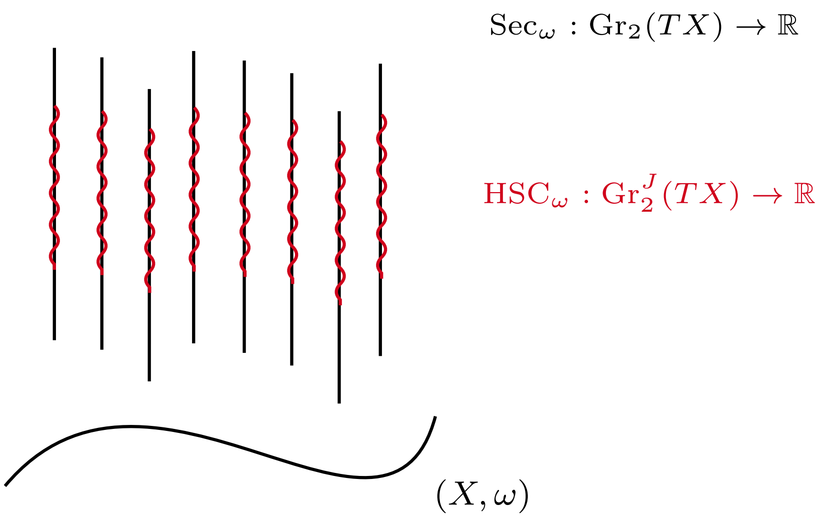
$$\mathrm{Sec}_\omega : \mathrm{Gr}_2(TX) \longrightarrow \mathbb{R},$$

on the Grassmannian of 2–planes in the tangent bundle of X.

Inside $\mathrm{Gr}_2(TX)$, we have a $\mathbb{P}^n$–bundle given by the 2–planes invariant under the complex structure J



The restriction of the sectional curvature to this $\mathbb{P}^n$–bundle defines the holomorphic sectional curvature.



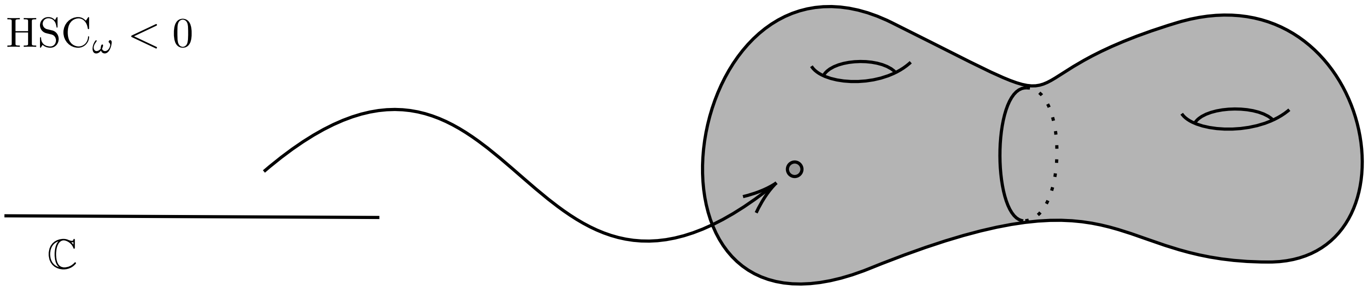
If R denotes the (Riemannian) curvature tensor of a Kähler metric ω, with complex structure J, the holomorphic sectional curvature is given by

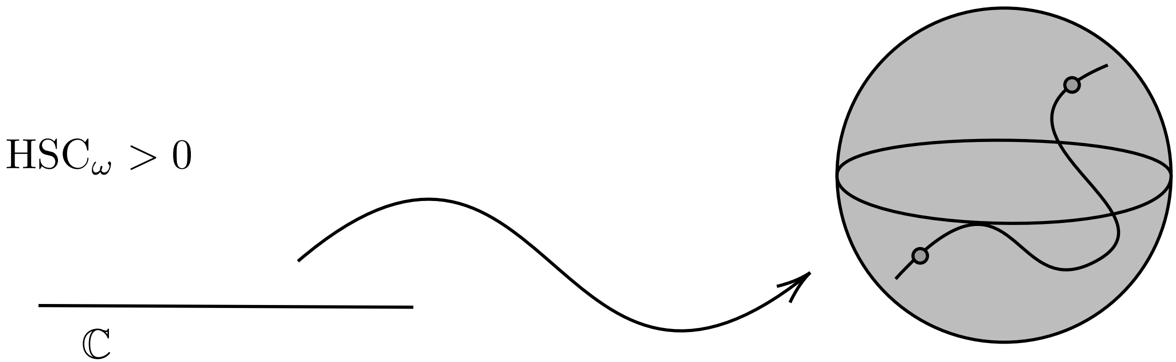
$$\mathrm{HSC}_\omega(u) \ := \ \frac{1}{|u|_\omega^4} R(u, Ju, u, Ju).$$

In terms of $(1,0)$–vectors $v \in T^{1,0}X$, $v = u - \sqrt{-1}Ju$ the holomorphic sectional curvature reads

$$\mathrm{HSC}_\omega(v) \ = \ \frac{1}{|v|_\omega^4} \sum_{i,j,k,\ell=1}^{n} R_{i\bar{j}k\bar{\ell}} v_i \overline{v}_j v_k \overline{v}_\ell.$$

The holomorphic sectional curvature controls the distortion of holomorphic maps.





A compact Kähler manifold (X, ω) with

(†) $\text{HSC}_\omega < 0$ is Kobayashi hyperbolic – all holomorphic maps $\mathbb{C} \to X$ are constant.

(†) $\text{HSC}_\omega > 0$ is rationally connected – any two points lie in the image of a rational curve $\mathbb{P}^1 \to X$.

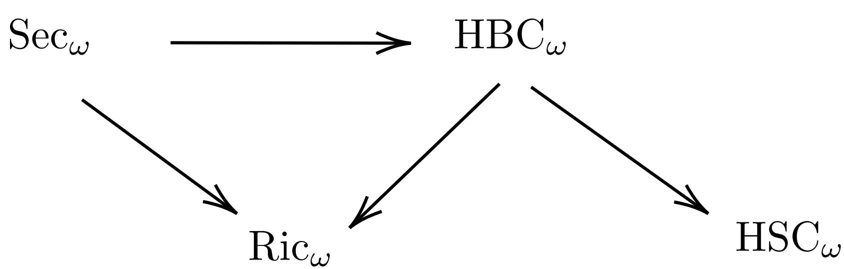
The holomorphic sectional curvature is in a similar place to the Ricci curvature in the curvature heirarchy.

They are both dominated by the holomorphic bisectional curvature, and both dominate the scalar curvature.

The holomorphic bisectional curvature HBC_ω of a Kähler metric is defined

$$\mathrm{HBC}_\omega(u,v) \;:=\; \frac{1}{|u|_\omega^2 |v|_\omega^2} R(u, Ju, v, Jv).$$

Clearly[2]:



[2] Arrows indicate curvature dominance

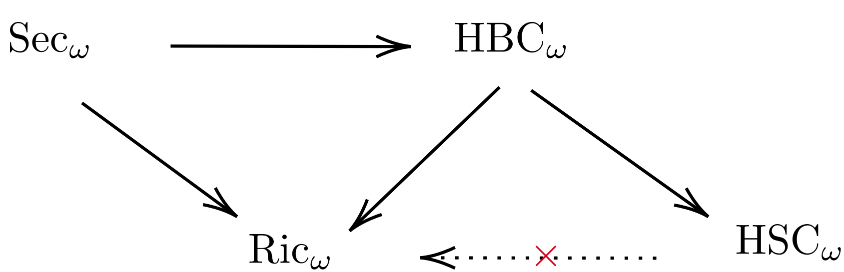
Constraints on HBC_ω are very restrictive:

(Mori, Siu–Yau solution of Frankel conjecture):

Compact Kähler with $\mathrm{HBC}_\omega > 0 \implies X \simeq_{\text{bihol.}} \mathbb{P}^n$.

The holomorphic sectional curvature does not dominate the Ricci curvature, however.

Hitchin's examples of Hodge metrics on Hirzebruch surfaces have $\mathrm{HSC}_\omega > 0$ but $\mathrm{Ric}_\omega \not> 0$.



The Wu–Yau theorem states the following curious relationship between the Ricci curvature and the holomorphic sectional curvature:

If (X, ω) is compact Kähler. Then

$$\mathrm{HSC}_{\omega} < 0 \implies \exists\ \omega_{\varphi} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi \text{ such that } \mathrm{Ric}_{\omega_{\varphi}} < 0.$$

In particular,

$$\mathrm{HSC}_{\omega} < 0 \implies K_X \text{ ample.}$$