MATH3405 – DIFFERENTIAL GEOMETRY

KYLE BRODER

Administration

Lecturers: Kyle Broder (weeks 1–3) and Ramiro Lafuente (from week 4).
Course coordinator: Ramiro Lafuente (office: 69–717).
Assessment:

- 5 assignments (10% each). Contribution 40% of final grade (choose 4 best marks among them). Due on weeks 3, 5, 7, 10, 13 (Friday 5pm).
- Final exam. Constribution 60% (See ECP).

References:

- Do Carmo – *Differential Geometry of Curves and Surfaces* (Main source).
- Klingenberg – *A course in differential geometry* (secondary, complementary resource).
- Ramiro's Lecture Notes. Kyle's lecture notes (the current notes).
- Peter Petersen – Classical Differential Geometry.
- Other textbooks, see ECP.

Support:

- Office hours: Wednesday 10am or by appointment. Kyle's office (69–705).
- Tutorials.
- Ed discussion forum (B6).

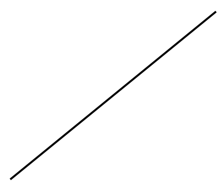
1. **Introduction**

Differential geometry aims to extend calculus (i.e., the theory of derivatives and integrals) that is very well-developed on Euclidean space $\mathbf{R}^n$ to more general spaces (e.g., the sphere $\mathbf{S}^2$ or torus $\mathbf{S}^1 \times \mathbf{S}^1$). This is essential for Einstein's general theory of relativity and can be important in many areas of applied mathematics, including statistics. This extension of calculus might be a means of studying problems that naturally emerge on spaces that are not $\mathbf{R}^n$ or may be a means of studying 'geometries' in general.

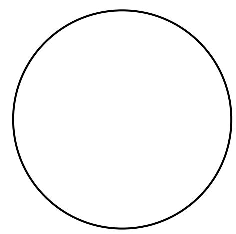
The first goal is to make sense of what "spaces" are reasonable objects for doing calculus on. In general, such objects are called *manifolds*, but we will not meet these objects until weeks 12 and 13 of the present course. Moreover, we will work with a slightly more general concept (namely, *immersed submanifolds*). This does not mean that all branches of (even differential) geometry confine themselves to spaces that are manifolds. For instance, algebraic geometers work with significantly more pathological objects such as *algebraic varieties*, *ringed spaces*, and *Hilbert schemes*.

2. **Curves**

We will first focus on the case of *curves*, which will serve as the “one-dimensional spaces”.[1] Let us first exhibit our expectations of what a “curve” is. Two clear examples are the straight line and the circle:



$\alpha(t) = (at, bt, ct) \in \mathbb{R}^3$



$\alpha(t) = (\cos t, \sin t, 0) \in \mathbb{R}^3$

Both of these examples are the images of maps $\alpha : I \to \mathbf{R}^3$ from an open interval $I = (a, b) \subseteq \mathbf{R}$ to $\mathbf{R}^3$. The map α is called a *parametrization*.

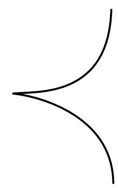
Definition 2.1. A *parametrized curve* is a map $\alpha : I \to \mathbf{R}^n$ from an interval $I \subseteq \mathbf{R}$. We call α a *parametrization*.

We say that a parametrized curve $\alpha : I \to \mathbf{R}^n$ is *smooth* if the parametrization α is $\mathcal{C}^\infty$–smooth, i.e., k–times continuously differentiable for all $k \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$. If the interval I is of the form $[a, b]$, $[a, b)$ or $(a, b]$, we understand smoothness to mean that there is a larger interval $\widetilde{I} \subset \mathbf{R}$ such that $I \subset \widetilde{I}$ and a smooth map $\widetilde{\alpha} : \widetilde{I} \to \mathbf{R}^n$ such that $\alpha = \widetilde{\alpha}|_I$. That is, α is the restriction of a smooth map defined on an open interval to I.

Example 2.2. The straight line given by the parametrization $\alpha : [0, 1] \to \mathbf{R}^2$, $\alpha(t) = (1 - t)\mathbf{p} + t\mathbf{q}$, for two vectors $\mathbf{p}, \mathbf{q} \in \mathbf{R}^2$ is a smooth curve. The circle of radius $r > 0$ given by $\alpha : [0, 2\pi] \to \mathbf{R}^2$, $\alpha(t) = (\cos(t), \sin(t))$ is a smooth curve. The extensions are the obvious ones.

Even if the parametrization is $\mathcal{C}^\infty$–smooth, the following example illustrates that singularities may still arise:

[1]Note that, at this point, no meaningful definition of space, let alone dimension, has been given.



$\alpha(t) = (t^2, t^3) \in \mathbf{R}^2$

The *cusp singularity* occurs at the point $t = 0$, corresponding to the point where $\alpha'(t) = (2t, 3t^2) = (0,0)$. We, therefore, make the following definition:

Definition 2.3. A parametrized curve $\alpha : I \to \mathbf{R}^3$ is said to be *regular* if α is $\mathcal{C}^\infty$–smooth and $\alpha'(t) \neq 0$ for all $t \in I$.

Example 2.4. The *lemniscate* is the parametrized curve $\alpha : \mathbf{R} \to \mathbf{R}^2$ defined by

$$\alpha(t) \ := \ \left(\frac{t(1+t^2)}{1+t^4}, \frac{t(1-t^2)}{1+t^4} \right).$$

If we write $x(t) := \frac{t(1+t^2)}{1+t^4}$ and $y(t) := \frac{t(1-t^2)}{1+t^4}$, determine the points where $x'(t) = 0$ and $y'(t) = 0$. Show that α is a regular curve and compute $|\alpha'(t)|$.

Remark 2.5. This provides an example of an *immersed submanifold* that is not an *embedded submanifold.*

We do not want to speak about things that are dependent on the way that we have written them down. [2] For instance, the matrices that we encounter in linear algebra are not mathematical objects – they are dependent on how we write them down. Namely, they are the manifestation of writing a linear operator in terms of a choice of basis. This is why linear algebra focuses so heavily on the theory of eigenvalues and eigenvectors – they allow us to determine properties of the underlying linear operator and are independent of any choice of basis.

In the present context, we do not want our "geometry" to be dependent on the choice of parametrization. For instance, consider the following parametrizations of the unit circle $\alpha : (0, 2\pi) \to \mathbf{R}^2$ given by $\alpha(t) = (\cos t, \sin t)$ and $\beta : (0, \pi) \to \mathbf{R}^2$ given by $\beta(t) = (\cos 2t, \sin 2t)$. We think of the parametrization β as tracing out the unit circle twice as fast as the parametrization α. We would think of these as defining the same "curve", with the only distinction coming from the parametrization (i.e., how they have been written down).

[2] As remarked by Misha Gromov, "Mathematics should be independent of the blackboard. "

Definition 2.6. Let $\alpha : I \to \mathbf{R}^3$ be a smooth parametrized curve. We say that a smooth parametrized curve $\beta : J \to \mathbf{R}^3$ is a *reparametrization* of α if $\beta = \alpha \circ \Phi$, where $\Phi : J \to I$ is an invertible smooth map with smooth inverse. If α is a regular parametrized curve, then we say that β is a *regular reparametrization* of α if $\beta = \alpha \circ \Phi$, where $\Phi : J \to I$ is an invertible smooth map with smooth inverse such that $\Phi'(t) > 0$ for all $t \in J$ and $(\Phi^{-1})'(t) > 0$ for all $t \in I$.

Heuristically, we think of a reparametrization as a change of variables.

Both reparametrization and regular reparametrization define an equivalence relation on the space of smooth parametrized curves and regular parametrized curves, respectively.

Definition 2.7. A *curve* is a parametrization equivalence class of smooth parametrized curves. A *regular curve* is a regular reparametrization equivalence class of regular parametrized curves.

Remark 2.8. A regular curve may admit a smooth parametrization that is not regular. For instance, $\alpha : [0,1] \to \mathbf{R}^2$ defined by $\alpha(t) = (t^2, t^2)$ parametrizes a line in $\mathbf{R}^2$. But $\alpha'(t) = (2t, 2t)$, which vanishes at $t = 0$. Hence, a curve is *regular* if there is *some* regular parametrization.

In the study of (regular) curves, we now want to understand when two curves that we encounter in the wild, which a priori may be given by very complicated parametrizations, are the same curve. This involves developing invariants that we can associate to the curves that are independent of the parametrization and are preferably computable.

An important invariant of the curve is its arclength.

Definition 2.9. Let $\alpha : I \to \mathbf{R}^3$ be a regular parametrized curve. The *length* of $\alpha(t)$ between two points $t_1, t_2 \in I$ is given by

$$\mathcal{L}(\alpha|_{[t_1,t_2]}) \ := \ \int_{t_1}^{t_2} |\alpha'(t)| dt.$$

Definition 2.10. We say that a regular parametrized curve $\alpha : I \to \mathbf{R}^3$ is *parametrized by arc-length* if $|\alpha'(t)| \equiv 1$ for all $t \in I$.

The arc-length parametrization is typically denoted by the parameter s.

Proposition 2.11. Any regular parametrized curve $\alpha : I \to \mathbf{R}^3$ can be reparametrized by its arc length.

Proof. Let $I = (a, b)$ for $a, b \in \mathbf{R}$ with $a < b$. Set $s(t) := \int_a^t |\alpha'(u)| du$. By the fundamental theorem of calculus, s is differentiable with $\frac{d}{dt} s(t) = |\alpha'(t)| > 0$, with the strict positivity following from the fact that α is regular. Let $J := s(I)$. Then $s : I \to J$ is a strictly increasing function, and therefore, injective. In particular, it is bijective onto its image $s(I) = J$. It is therefore invertible with inverse $\Phi : J \to I$. By the inverse function theorem, Φ is differentiable. Set $\beta(s) := \alpha(\Phi(s))$. Then

$$\frac{d}{ds}\beta(s) \;=\; \alpha'(\Phi(s))\frac{d}{ds}\Phi(s) \;=\; \alpha'(\Phi(s))\frac{1}{s'(\Phi(s))},$$

where the first equality is a consequence of the chain rule, and the latter follows from the fact that $\Phi = s^{-1}$. Hence, $|\beta'(s)| = |\alpha'(\Phi(s))|/|s'(\Phi(s))| = |\alpha'(\Phi(s))|/|\alpha'(\Phi(s))| = 1$. □

Example 2.12. Let $\alpha : [0, 2\pi] \to \mathbf{R}^2$ be given by $\alpha(t) = (r\cos t, r\sin t)$ for $r > 0$. Then $\alpha'(t) = (-r\sin t, r\cos t)$, and hence, $|\alpha'(t)| = r$. Then $s(t) = \int_0^t r du = rt$, and therefore, $\phi(s) = s/r$. In particular, the arc length parametrization is

$$\beta(s) = (r\cos(s/r), r\sin(s/r)).$$

Remark 2.13. In practice, the arc length parametrization is very hard to compute. Even the arc length of an ellipse is highly non-trivial, which lead to the theory of abelian integrals, which later became Hodge theory.

Proposition 2.14. Let $\alpha : I \to \mathbf{R}^3$ be a regular parametrized curve. The arc length of the curve $\mathcal{C} := \alpha(I)$ is independent of the choice of parametrization. In particular, arc length is an intrinsic property.

Proof. Let $\alpha : I \to \mathbf{R}^3$ be a regular parametrized curve. Let $\beta : J \to \mathbf{R}^3$ be another parametrization with regular reparametrization $\Phi : J \to I$. Then

$$\mathcal{L}(\alpha(I)) \;=\; \int_I |\alpha'(t)| dt, \qquad \mathcal{L}(\beta(J)) \;=\; \int_J |\beta'(s)| ds.$$

Observe that $t = \Phi(s)$, and therefore, $dt = \frac{d\Phi}{ds} ds$. Further, $\beta(s) = \alpha(\Phi(s))$, and therefore,

$$\beta'(s) \;=\; \frac{d\Phi}{ds}\alpha'(\Phi(s)) \;=\; \frac{d\Phi}{ds}\alpha'(t).$$

Hence,

$$\mathcal{L}(\beta(J)) \;=\; \int_J |\beta'(s)| ds \;=\; \int_{\Phi^{-1}(J)} \left| \frac{d\Phi}{ds}\alpha'(t) \right| \frac{dt}{\frac{d\Phi}{ds}} \;=\; \int_I |\alpha'(t)| dt \;=\; \mathcal{L}(\alpha(I)).$$

□

Remark 2.15. We will see that in higher dimensions, the role of the arc length is played by a Riemannian metric and the arc length parametrization is given by the exponential map.

3. **Invariants of a Regular Curve**

Now that we have some understanding of the objects that we want to study – namely, regular curves $\mathcal{C}$ in $\mathbf{R}^n$, we want to determine when two curves are the same and in particular when they are different. We have already seen one such 'invariant' – the arc length. Of course, just because two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same arc length, does not mean that they are the same curve.

Definition 3.1. Let $\alpha : I \to \mathbf{R}^3$ be a regular curve, parametrized by arc length. The *curvature of* α is the function $\kappa : I \to \mathbf{R}$ defined by $\kappa(s) := |\alpha''(s)|$.

The curvature $K(s)$ measures how fast the curve pulls away from the tangent line at $\alpha(s)$. To understand the geometry of a curve $\mathcal{C}$ in $\mathbf{R}^3$, let $\alpha : I \to \mathbf{R}^3$ be a regular arc-length parametrization. Since $\mathcal{C}$ is regular, the tangent vector $\mathbf{T}(s) := \alpha'(s)$ is a non-zero vector, tangent to the curve at each point $\alpha(s) \in \mathcal{C}$. For the second derivative, we have the following:

Proposition 3.2. Let $\alpha : I \to \mathbf{R}^3$ be a regular arc-length parametrization. Then $\mathbf{T}(s) := \alpha'(s)$ and $\alpha''(s)$ are orthogonal, with respect to the Euclidean inner product on $\mathbf{R}^3$.

Proof. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product on $\mathbf{R}^3$. Since α is a regular arc length parametrization, we have $|\alpha'(s)|^2 = \langle \alpha'(s), \alpha'(s) \rangle = 1$. Differentiating this equation, we see that

$$\begin{aligned} 0 \ = \ \frac{d}{ds}\langle t(s), t(s) \rangle \ &= \ \frac{d}{ds}\langle \alpha'(s), \alpha'(s) \rangle \\ &= \ \langle \alpha''(s), \alpha'(s) \rangle + \langle \alpha'(s), \alpha''(s) \rangle \ = \ 2\langle \alpha''(s), \alpha'(s) \rangle. \end{aligned}$$

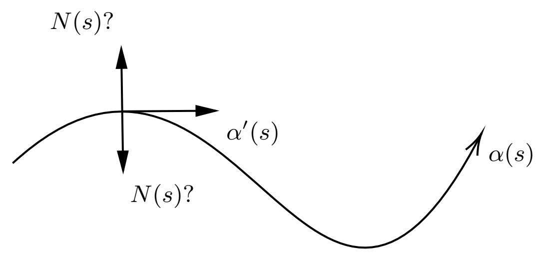
□

Definition 3.3. Let $\alpha : I \to \mathbf{R}^3$ be a regular curve parametrized by arc length. The *normal vector* $\mathbf{N}(s) \in \mathbf{R}^3$ is defined by

$$\alpha''(s) \ = \ K(s)\mathbf{N}(s),$$

where $K(s) > 0$ is the curvature of α.

Remark 3.4. In the case of plane curves, i.e., curves $\alpha : I \to \mathbf{R}^2$, we can speak about the sign of the curvature. Proposition 3.2 implies that $\alpha''(s)$ is othogonal to the curve $\mathcal{C}$ at each point $\alpha(s)$. But there remains an ambiguity given by *orientation*. At present, it is not clear which of the following (or both) are permitted:



Let us denote by $e_1 = (1,0)$ and $e_2 = (0,1)$ denote the standard basis of $\mathbf{R}^2$.

Definition 3.5. Let $\mathcal{B} := \{v_1, v_2\}$ be a basis of $\mathbf{R}^2$. Let $A : \mathbf{R}^2 \to \mathbf{R}^2$ be the invertible matrix that maps the basis $\{e_1, e_2\}$ onto $\mathcal{B}$. We say that the basis $\mathcal{B}$ is *positively oriented* (or has the same orientation) if the determinant of this matrix is positive, i.e., $\det(A) > 0$.

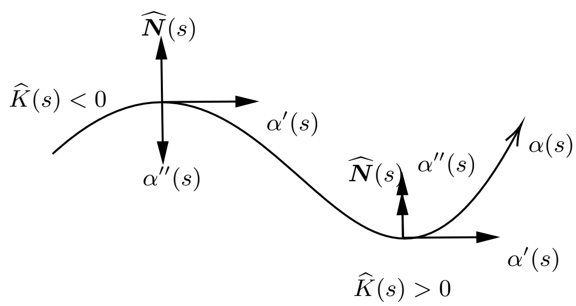
Example 3.6. Let $u = (-1,1)$ and $v = (1,-1)$ be two vectors in $\mathbf{R}^2$. The basis $\{(1,1), u\}$ is positively oriented, and $\{(1,1), v\}$ is negatively oriented.

Definition 3.7. Let $\mathcal{C}$ be a regular curve with $\alpha : I \to \mathbf{R}^3$ a regular arc length parametrization. We define the (positively oriented) *normal vector* $\widehat{\mathbf{N}}(s)$ to be the unit vector such that $\{\mathbf{T}(s), \widehat{\mathbf{N}}(s)\}$ is a positively oriented orthonormal basis of $\mathbf{R}^2$.

Definition 3.8. Let $\mathcal{C}$ be a regular curve with $\alpha : I \to \mathbf{R}^3$ a regular arc length parametrization. The *signed curvature* $\widehat{K}(s)$ is defined uniquely by the equation

$$\alpha''(s) \;=\; \widehat{K}(s)\widehat{\mathbf{N}}(s).$$

The *curvature* (or *unsigned curvature*) $K(s)$ is the Euclidean norm of the signed curvature, i.e., $K(s) := |\widehat{K}(s)| = |\alpha''(s)|$.



Example 3.9. We saw that an arc length parametrization for the circle of radius r is given by $\beta : [0, 2\pi] \to \mathbf{R}^2$,

$$\beta(s) = \left(r \cos\left(\frac{s}{r}\right), r \sin\left(\frac{s}{r}\right)\right).$$

The derivatives are

$$\beta'(s) = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\right),$$
$$\beta''(s) = \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right)\right).$$

Hence, the curvature is

$$\kappa(s) = |\beta''(s)| = \sqrt{\left(-\frac{1}{r}\cos\left(\frac{s}{r}\right)\right)^2 + \left(-\frac{1}{r}\sin\left(\frac{s}{r}\right)\right)^2} = \sqrt{\frac{1}{r^2}} = \frac{1}{r}.$$

While the (signed or unsigned) curvature is an important invariant of a regular curve, it will not be sufficient to determine the geometry of the curve completely. To produce a further invariant of the curve, we need to complete the orthonormal basis $\{\mathbf{T}(s), \mathbf{N}(s)\}$ to an orthonormal basis of $\mathbf{R}^3$. In particular, we need to produce a unit vector that is orthogonal to both $\mathbf{T}(s)$ and $\mathbf{N}(s)$. An easy means of doing this is by exploiting the cross product $\times$.

Definition 3.10. Let $\mathcal{C}$ be a regular curve with $\alpha : I \to \mathbf{R}^3$ a regular arc length parametrization. Let $\mathbf{T}(s)$ and $\mathbf{N}(s)$ denote the tangent and (positively oriented) normal vector to $\mathcal{C}$ at $\alpha(s)$. We define the *binormal vector* by $\mathbf{B}(s) := \mathbf{T}(s) \times \mathbf{N}(s)$.

Note that since

$$|\mathbf{B}(s)| = |\mathbf{T}(s) \times \mathbf{N}(s)| = |\mathbf{T}(s)| \cdot |\mathbf{T}(s)| \cdot |\sin(\vartheta)| = |\sin(\pi/2)| = 1,$$

it follows that $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is an orthonormal basis of $\mathbf{R}^3$.

Definition 3.11. Let $\mathcal{C}$ be a regular curve with $\alpha : I \to \mathbf{R}^3$ a regular arc length parametrization. We call the orthonormal basis $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ of $\mathbf{R}^3$ a *Frenet frame* for $\mathcal{C}$.

From the binormal vector $\mathbf{B}(s)$, we observe that the derivative $\mathbf{B}'(s)$ lies in the span of $\mathbf{N}(s)$. Indeed, it suffices to show that $\mathbf{B}'(s)$ is othogonal to both $\mathbf{T}(s)$ and $\mathbf{B}(s)$. Differentiating $|\mathbf{B}(s)|^2 = 1$, we see that

$$0 \ = \ \frac{d}{ds}\langle \mathbf{B}(s), \mathbf{B}(s)\rangle \ = \ 2\langle \mathbf{B}'(s), \mathbf{B}(s)\rangle.$$

This implies that $\mathbf{B}'(s)$ is orthogonal to $\mathbf{B}(s)$. Further, since $\mathbf{B}(s)$ and $\mathbf{T}(s)$ are orthogonal, differentiating $\langle \mathbf{B}(s), \mathbf{T}(s)\rangle = 0$ yields

$$\begin{aligned} 0 &= \langle \mathbf{B}'(s), \mathbf{T}(s)\rangle + \langle \mathbf{B}(s), \mathbf{T}'(s)\rangle \\ &= \langle \mathbf{B}'(s), \mathbf{T}(s)\rangle + K(s)\langle \mathbf{B}(s), \mathbf{N}(s)\rangle \\ &= \langle \mathbf{B}'(s), \mathbf{T}(s)\rangle, \end{aligned}$$

which proves the claim.

From here, we may define a new invariant of a regular parametrized curve:

Definition 3.12. Let $\mathcal{C}$ be a regular curve with $\alpha : I \to \mathbf{R}^3$ a regular arc length parametrization. We define the *torsion* $\tau(s)$ to be the unique function that satisfies

$$\mathbf{B}'(s) \ = \ \tau(s)\mathbf{N}(s).$$

Remark 3.13. The binormal $\mathbf{B}(s)$ determines the so-called *osculating plane*, defined by the span of $\mathbf{T}(s)$ and $\mathbf{N}(s)$. Hence, the measures the extent to which the curve deviates from its osculating plane. Note that the span of $\mathbf{N}(s)$ and $\mathbf{B}(s)$ is called the *normal plane*, and the span of $\mathbf{T}(s)$ and $\mathbf{B}(s)$ is called the *rectifying plane*.

Theorem 3.14. (Frenet Formulae). Let $\alpha : I \to \mathbf{R}^3$ be a regular curve parametrized by arc length with non-zero curvature. Denote by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ the Frenet frame at each point $s \in I$. Then

$$\begin{aligned} \frac{d\mathbf{T}}{ds} &= K(s)\mathbf{N}(s), \\ \frac{d\mathbf{N}}{ds} &= -K(s)\mathbf{T}(s) - \tau(s)\mathbf{B}(s), \\ \frac{d\mathbf{B}}{ds} &= \tau(s)\mathbf{N}(s). \end{aligned}$$

Proof. The first and third equations hold by definition. Hence, we need only prove the second formula. To this end, since $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is an orthonormal basis of $\mathbf{R}^3$ for each $s \in I$, we may write

$$\frac{d\mathbf{N}}{ds} = \left\langle \mathbf{N}'(s), \mathbf{T}(s) \right\rangle \mathbf{T}(s) + \left\langle \mathbf{N}'(s), \mathbf{N}(s) \right\rangle \mathbf{N}(s) + \left\langle \mathbf{N}'(s), \mathbf{B}(s) \right\rangle \mathbf{B}(s).$$

Differentiating $\langle \mathbf{N}(s), \mathbf{T}(s) \rangle = 0$, shows that

$$0 = \langle \mathbf{N}'(s), \mathbf{T}(s) \rangle + \langle \mathbf{N}(s), \mathbf{T}'(s) \rangle = \langle \mathbf{N}'(s), \mathbf{T}(s) \rangle + K(s) \langle \mathbf{N}(s), \mathbf{N}(s) \rangle.$$

Since $\|\mathbf{N}(s)\|^2 = 1$, it follows that $\langle \mathbf{N}'(s), \mathbf{T}(s) \rangle \mathbf{T}(s) = -K(s)\mathbf{T}(s)$. Differentiating $\|\mathbf{N}(s)\|^2 = 1$ shows that $\langle \mathbf{N}'(s), \mathbf{N}(s) \rangle \mathbf{N}(s) = 0$. Finally, differentiating $\langle \mathbf{N}(s), \mathbf{B}(s) \rangle = 0$ shows that

$$\left\langle \mathbf{N}'(s), \mathbf{B}(s) \right\rangle \mathbf{B}(s) = -\tau(s)\mathbf{B}(s),$$

which completes the proof. □

4. **Rigid Motions**

The main theorem on the geometry of curves is that for regular curves with non-zero curvature, the curvature and torsion almost determine the curve uniquely. It does not precisely determine the curve uniquely, but it does determine the curve uniquely, up to a rigid motion of the ambient $\mathbf{R}^n$. To be precise, let us remind ourselves:

Definition 4.1. Let $d : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ denote the Euclidean distance function, given by

$$d(u, v) := |u - v|^2.$$

A map $f : \mathbf{R}^3 \to \mathbf{R}^3$ is called a *rigid motion* if it is an isometry with respect to d, i.e.,

$$d(f(u), f(v)) = d(u, v)$$

for all $u, v \in \mathbf{R}^3$.

Example 4.2. Let $w \in \mathbf{R}^3$ be a fixed vector. Define a map $T_w : \mathbf{R}^3 \to \mathbf{R}^3$ by $T_w(u) := u+w$. Observe that T_v is a rigid motion:

$$\begin{aligned} d(T_w(u), T_w(v)) &= |T_w(u) - T_w(v)| \\ &= |u + w - (v + w)| = |u - v| = d(u, v). \end{aligned}$$

In particular, translations are rigid motions.

We can give a complete description of the rigid motions on $\mathbf{R}^n$. To this end, we recall from our first course in linear algebra:

Definition 4.3. Let $f : V \to W$ be a map between two (real, finite-dimensional) vector spaces. We say that f is *linear* if

(i) $f(u + v) = f(u) + f(v)$ for all $u, v \in V$.
(ii) $f(\lambda v) = \lambda f(v)$ for all $v \in V$ and $\lambda \in \mathbf{R}$.

Remark 4.4. A matrix is the result of writing a linear map down with respect to a choice of basis. That is, if $e_1, ..., e_n$ is the standard basis of $\mathbf{R}^n$, we obtain a matrix $A = (A_{ij})$ from a linear map $T : \mathbf{R}^n \to \mathbf{R}^n$ from the formula

$$T(e_j) = \sum_{i=1}^{n} A_{ij} e_i.$$

Conversely, for any matrix A, we obtain a linear map $L_A : \mathbf{R}^n \to \mathbf{R}^n$ given by $L_A(v) := Av$.

The linear rigid motions (i.e., rigid motions that are also linear maps) are given by the class of orthogonal matrices:

Definition 4.5. Let $A \in \mathbf{R}^{n\times n}$ be a square matrix. We say that A is *orthogonal* if $A^T A = \mathrm{id}$, where A^T denotes the transpose of A and $\mathrm{id} : \mathbf{R}^n \to \mathbf{R}^n$ is the identity map.

Proposition 4.6. The set of orthogonal $n \times n$ matrices forms a group that we denote by $\mathrm{O}(n)$.

5. Fundamental Theorem of Curves

In the present section, we restrict our attention to regular curves $\alpha : I \to \mathbf{R}^3$ parametrized by arc length and with non-zero curvature. The main theorem is the following:

Theorem. Let $I \subseteq \mathbf{R}$ be an interval. Let $K, \tau : I \to \mathbf{R}$ be arbitrary smooth functions with $K(s) > 0$ for all $s \in I$. Then

(i) there is a regular curve $\alpha : I \to \mathbf{R}^3$ of unit speed whose curvature and torsion are $K(s)$ and $\tau(s)$.

(ii) If $\widetilde{\alpha} : I \to \mathbf{R}^3$ is another curve with curvature $K(s)$ and torsion $\tau(s)$, then there is an orientation-preserving rigid motion $f : \mathbf{R}^3 \to \mathbf{R}^3$ such that $\widetilde{\alpha}(s) = f(\alpha(s))$ for all $s \in I$.

Proof of (ii) – Uniqueness. Let $\alpha = \alpha(s)$ and $\widetilde{\alpha} = \widetilde{\alpha}(s)$ be two curves such that $K(s) = \widetilde{K}(s)$ and $\tau(s) = \widetilde{\tau}(s)$, for all $s \in I$. Let $\{\mathbf{T}(s_0), \mathbf{N}(s_0), \mathbf{B}(s_0)\}$ and $\{\widetilde{\mathbf{T}}(s_0), \widetilde{\mathbf{N}}(s_0), \widetilde{\mathbf{B}}(s_0)\}$ denote the respective Frenet frames for α and $\widetilde{\alpha}$ and $\alpha(s_0)$ and $\widetilde{\alpha}(s_0)$. There is a rigid motion that takes $\widetilde{\alpha}(s_0)$ to $\alpha(s_0)$ and $\{\widetilde{\mathbf{T}}(s_0), \widetilde{\mathbf{N}}(s_0), \widetilde{\mathbf{B}}(s_0)\}$ to $\{\mathbf{T}(s_0), \mathbf{N}(s_0), \mathbf{B}(s_0)\}$. Hence, upon applying this rigid motion, we may assume that $\widetilde{\alpha}(s_0) = \alpha(s_0)$ and that the Frenet frames coincide at s_0.

The Frenet formulae tell us that

$$\begin{aligned} \frac{d\mathbf{T}}{ds} &= K(s)\mathbf{N}(s) \\ \frac{d\mathbf{N}}{ds} &= -K(s)\mathbf{T}(s) - \tau(s)\mathbf{B}(s), \\ \frac{d\mathbf{B}}{ds} &= \tau(s)\mathbf{N}(s), \end{aligned}$$

and

$$\begin{aligned} \frac{d\widetilde{\mathbf{T}}}{ds} &= K(s)\widetilde{\mathbf{N}}(s) \\ \frac{d\widetilde{\mathbf{N}}}{ds} &= -K(s)\widetilde{\mathbf{T}}(s) - \tau(s)\widetilde{\mathbf{B}}(s), \\ \frac{d\widetilde{\mathbf{B}}}{ds} &= \tau(s)\widetilde{\mathbf{N}}(s), \end{aligned}$$

with $\mathbf{T}(s_0) = \widetilde{\mathbf{T}}(s_0)$, $\mathbf{N}(s_0) = \widetilde{\mathbf{N}}(s_0)$, $\mathbf{B}(s_0) = \widetilde{\mathbf{B}}(s_0)$.

Let us compute

$$\begin{aligned} \frac{d}{ds}\|\mathbf{T}(s) - \widetilde{\mathbf{T}}(s)\|^2 &= \frac{d}{ds}\langle \mathbf{T}(s) - \widetilde{\mathbf{T}}(s), \mathbf{T}(s) - \widetilde{\mathbf{T}}(s)\rangle \\ &= 2\langle \mathbf{T}'(s) - \widetilde{\mathbf{T}}'(s), \mathbf{T}(s) - \widetilde{\mathbf{T}}(s)\rangle \\ &= 2\langle \mathbf{T}'(s), \mathbf{T}(s)\rangle - 2\langle \widetilde{\mathbf{T}}'(s), \mathbf{T}(s)\rangle - 2\langle \mathbf{T}'(s), \widetilde{\mathbf{T}}(s)\rangle + \langle \widetilde{\mathbf{T}}'(s), \widetilde{\mathbf{T}}(s)\rangle \\ &= -2\langle \widetilde{\mathbf{T}}'(s), \mathbf{T}(s)\rangle - 2\langle \mathbf{T}'(s), \widetilde{\mathbf{T}}(s)\rangle \\ &= -2K(s)\langle \widetilde{\mathbf{N}}(s), \mathbf{T}(s)\rangle - 2K(s)\langle \mathbf{N}(s), \widetilde{\mathbf{T}}(s)\rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{d}{ds}\|\mathbf{N}(s) - \widetilde{\mathbf{N}}(s)\|^2 &= \frac{d}{ds}\langle \mathbf{N}(s) - \widetilde{\mathbf{N}}(s), \mathbf{N}(s) - \widetilde{\mathbf{N}}(s)\rangle \\ &= 2\langle \mathbf{N}'(s), \mathbf{N}(s)\rangle - 2\langle \widetilde{\mathbf{N}}'(s), \mathbf{N}(s)\rangle - 2\langle \mathbf{N}'(s), \widetilde{\mathbf{N}}(s)\rangle + \langle \widetilde{\mathbf{N}}'(s), \widetilde{\mathbf{N}}(s)\rangle \\ &= -2\langle \widetilde{\mathbf{N}}'(s), \mathbf{N}(s)\rangle - 2\langle \mathbf{N}'(s), \widetilde{\mathbf{N}}(s)\rangle \\ &= 2K(s)\langle \widetilde{\mathbf{T}}(s), \mathbf{N}(s)\rangle + 2\tau(s)\langle \widetilde{\mathbf{B}}(s), \mathbf{N}(s)\rangle \\ &\quad +2K(s)\langle \mathbf{T}(s), \widetilde{\mathbf{N}}(s)\rangle + 2\tau(s)\langle \mathbf{B}(s), \widetilde{\mathbf{N}}(s)\rangle. \end{aligned}$$

Again,

$$\begin{aligned} \frac{d}{ds}\|\mathbf{B}(s) - \widetilde{\mathbf{B}}(s)\|^2 &= \frac{d}{ds}\langle \mathbf{B}(s) - \widetilde{\mathbf{B}}(s), \mathbf{B}(s) - \widetilde{\mathbf{B}}(s)\rangle \\ &= 2\langle \mathbf{B}'(s), \mathbf{B}(s)\rangle - 2\langle \widetilde{\mathbf{B}}'(s), \mathbf{B}(s)\rangle - 2\langle \mathbf{B}'(s), \widetilde{\mathbf{B}}(s)\rangle + \langle \widetilde{\mathbf{B}}'(s), \widetilde{\mathbf{B}}(s)\rangle \\ &= -2\tau(s)\langle \widetilde{\mathbf{N}}(s), \mathbf{B}(s)\rangle - 2\tau(s)\langle \mathbf{N}(s), \widetilde{\mathbf{B}}(s)\rangle. \end{aligned}$$

Let now

$$\mathcal{Q}(s) := \frac{1}{2}\|\mathbf{T}(s) - \widetilde{\mathbf{T}}(s)\|^2 + \frac{1}{2}\|\mathbf{N}(s) - \widetilde{\mathbf{N}}(s)\|^2 + \frac{1}{2}\|\mathbf{B}(s) - \widetilde{\mathbf{B}}(s)\|^2.$$

Combining the previous computations, we see that

$$\begin{aligned}\frac{d}{ds}\mathcal{Q}(s) &= -K\langle\widetilde{\mathbf{N}},\mathbf{T}\rangle - K\langle\mathbf{N},\widetilde{\mathbf{T}}\rangle \\ &\quad +K\langle\widetilde{\mathbf{T}},\mathbf{N}\rangle + \tau\langle\widetilde{\mathbf{B}},\mathbf{N}\rangle + K\langle\mathbf{T},\widetilde{\mathbf{N}}\rangle + \tau\langle\mathbf{B},\widetilde{\mathbf{N}}\rangle - \tau\langle\widetilde{\mathbf{N}},\mathbf{B}\rangle - \tau\langle\mathbf{N},\widetilde{\mathbf{B}}\rangle \\ &= 0.\end{aligned}$$

for all $s \in I$. Hence, $\mathcal{Q}$ is constant, and since it vanishes for s_0, it vanishes identically. It follows that the Frenet frames coincide for all $s \in I$. Sine

$$\alpha'(s) = \mathbf{T}(s) = \widetilde{\mathbf{T}}(s) = \widetilde{\alpha}'(s),$$

we see that $\frac{d}{ds}(\alpha - \widetilde{\alpha})(s) = 0$ for all $s \in I$. Hene, $\alpha(s) = \widetilde{\alpha}(s) + c$. Since $\alpha(s_0) = \widetilde{\alpha}(s_0)$, we have $c = 0$.

Proof of (i) – Existence. The crux of the existence argument is to interpret the above theorem as an initial value problem for a system of differential equations, using the Frenet formulae, and apply the following theorem due to Picard and Lindelöf:

Theorem. (Picard–Lindelöf). Given initial conditions $(\xi_1)_0, ..., (\xi_9)_0$, there is an open interval $J \subset I$ containing s_0 and a unique differentiable mapping $\alpha : J \to \mathbf{R}^9$ with $\alpha(s_0) = ((\xi_1)_0, ..., (\xi_9)_0)$, and $\alpha'(s) = (f_1, ..., f_9)$, where each $f_1, ..., f_9$ is calculated in $(s, \alpha(s)) \in J \times \mathbf{R}^9$. Further, if the system is linear, $J = I$.

Observe that the Frenet equations

$$\begin{aligned}\frac{d\mathbf{T}}{ds} &= K(s)\mathbf{N}(s) \\ \frac{d\mathbf{N}}{ds} &= -K(s)\mathbf{T}(s) - \tau(s)\mathbf{B}(s) \\ \frac{d\mathbf{B}(s)}{ds} &= \tau(s)\mathbf{N}(s)\end{aligned}$$

can be understood as a system of differential equations on $I \times \mathbf{R}^9$:

$$\begin{aligned}\frac{d\xi_1}{ds} &= f_1(s, \xi_1, ..., \xi_9), \\ &\vdots \\ \frac{d\xi_9}{ds} &= f_9(s, \xi_1, ..., \xi_9),\end{aligned}$$

where $\mathbf{T} = (\xi_1, \xi_2, \xi_3)$, $\mathbf{N} = (\xi_4, \xi_5, \xi_6)$, and $\mathbf{B} = (\xi_7, \xi_8, \xi_9)$, and the f_i are linear functions (with coefficients depending on s).

From the Picard–Lindelöf theorem, given a Frenet frame $\{\mathbf{T}_0, \mathbf{N}_0, \mathbf{B}_0\}$, we have a family of functions $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$, for $s \in I$, with $\mathbf{T}(s_0) = \mathbf{T}_0$, $\mathbf{N}(s_0) = \mathbf{N}_0$, $\mathbf{B}(s_0) = \mathbf{B}_0$. We will show that $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ remains orothonormal for all $s \in I$. To this end, we compute

$$\begin{aligned}
\frac{d}{ds}\langle \mathbf{T}(s), \mathbf{N}(s)\rangle &= K(s)\langle \mathbf{N}(s), \mathbf{N}(s)\rangle - K(s)\langle \mathbf{T}(s), \mathbf{T}(s)\rangle - \tau(s)\langle \mathbf{T}(s), \mathbf{B}(s)\rangle \\
&= K(s)\|\mathbf{N}(s)\|^2 - K(s)\|\mathbf{T}(s)\|^2 \\
&= K(s) - K(s) \ = \ 0 \\
\frac{d}{ds}\langle \mathbf{T}(s), \mathbf{B}(s)\rangle &= K(s)\langle \mathbf{N}(s), \mathbf{B}(s)\rangle + \tau(s)\langle \mathbf{T}(s), \mathbf{N}(s)\rangle \ = \ 0, \\
\frac{d}{ds}\langle \mathbf{N}(s), \mathbf{B}(s)\rangle &= -K(s)\langle \mathbf{T}(s), \mathbf{B}(s)\rangle - \tau(s)\langle \mathbf{B}(s), \mathbf{B}(s)\rangle + \tau(s)\langle \mathbf{N}(s), \mathbf{N}(s)\rangle, \\
&= -\tau(s)\|\mathbf{B}(s)\|^2 + \tau(s)\|\mathbf{N}(s)\|^2 \ = \ 0 \\
\frac{d}{ds}\langle \mathbf{T}(s), \mathbf{T}(s)\rangle &= 2K(s)\langle \mathbf{T}(s), \mathbf{N}(s)\rangle \ = \ 0 \\
\frac{d}{ds}\langle \mathbf{N}(s), \mathbf{N}(s)\rangle &= -2K(s)\langle \mathbf{N}(s), \mathbf{T}(s)\rangle - 2\tau(s)\langle \mathbf{N}(s), \mathbf{B}(s)\rangle \ = \ 0 \\
\frac{d}{ds}\langle \mathbf{B}(s), \mathbf{B}(s)\rangle &= 2\tau(s)\langle \mathbf{B}(s), \mathbf{N}(s)\rangle \ = \ 0.
\end{aligned}$$

From the moving frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ we obtain a curve by setting

$$\alpha(s) \ := \ \int_I \mathbf{T}(t)dt.$$

6. Preliminaries from Multivariable Calculus

Theorem. (Inverse function theorem). Let $f : \mathbf{R} \to \mathbf{R}$ be a $\mathcal{C}^1$ function with $f'(p) \neq 0$ for some point $p \in \mathbf{R}$. Then f is invertible in a neighborhood of p with the derivative of the inverse $g := f^{-1}$ at $q = f(p)$ given by $g'(q) = 1/f'(a)$.