Chapter 1

Exponentials and Logarithms.

1.1 Revision of Index Laws

Let us begin by recalling the following index laws.

† $a^x \cdot a^y = a^{x+y}$.

† $a^x / a^y = a^{x-y}$.

† $(a^x)^y = a^{x \cdot y}$.

† $a^0 = 1$.

Example 2.1.1. Find the values of x such that $3^{2x-1} = 81$.

Proof. Recall that $81 = 3^4$. Therefore, we see that

$$\begin{aligned} 3^{2x-1} = 81 \quad &\implies \quad 3^{2x-1} = 3^4 \\ &\implies \quad 2x - 1 = 4 \\ &\implies \quad 2x = 5 \\ &\implies \quad x = \frac{5}{2}. \end{aligned}$$

□

Exercises

Q1. Simplify the following expressions

a. $$3x^3y \cdot 4x^6y^5.$$

b. $$5x^{\frac{1}{2}}y^4 \cdot \frac{1}{4}x^{-5}y^{\frac{4}{3}}.$$

c. $$2x^3y^4 \cdot 4x^9y^{-13}.$$

d. $$\frac{3x^3y^4}{4x^2y^{10}}.$$

e. $$\frac{2(x^3y^2)^4}{6\sqrt{x^4y^8}}.$$

f. $$\frac{x^2+y^2}{x^{-2}+y^{-2}}.$$

g. $$\frac{x^3+y^{-3}}{x^2-y^{-4}}.$$

h. $$\frac{x+y}{x^{-1}y^{-1}}.$$

Q2. Solve the following equations for $x \in \mathbb{R}$.

a. $3^x = 9$.

b. $81^x = 27$.

c. $5^{-x} = \frac{1}{625}$.

d. $16^x = 1024$.

e. $625^x = 5$.

f. $4^x = 256$.

g. $2^{3x-6} = 8$.

h. $2^{x^2-5x} = \frac{1}{64}$.

Q3. Solve the following equations for $x \in \mathbb{R}$.

a. $2^{3x} \cdot 4^{4x-1} = 16$.

b. $3^{x-6} \cdot 9^{4x+12} = 81$.

c. $2^{x+1} \cdot 16^{4-x} = 1$.

d. $3^{4x+15} \cdot \frac{1}{3^{15-x}} = 1$.

Q4. Solve the following equations for $x \in \mathbb{R}$.

a. $9^x - 12 \cdot 3^x + 27 = 0$.

b. $4^x - 5 \cdot 2^x + 6 = 0$.

c. $9^x + 2 \cdot 3^x + 1 = 0$.

d. $16^x - 5 \cdot 4^x + 4 = 0$.

Q5. (Dr. Lloyd Gunatilake). Simplify the following expressions.

a. $$\frac{3^n - 3^{n+3}}{2 \cdot 3^{n-1} - 3^{n+1}}.$$

b. $$\frac{12^n - 6^n}{2^n - 1}.$$

Q6. (Dr. Lloyd Gunatilake). Simplify each of the following expressions.

a.
$$\frac{x^{-1}}{x^{-1}+1} - \frac{x^{-1}+1}{x^{-1}-1}.$$

b.
$$\frac{x\sqrt{3x-2}}{\sqrt{2x-1}} - \frac{x\sqrt{2x-1}}{\sqrt{3x-2}}.$$

c.
$$\frac{3\sqrt{x}}{\frac{5}{\sqrt{x}}+2} - \frac{2\sqrt{x}}{\frac{5}{\sqrt{x}}-2}.$$

1.2 Logarithm Laws

The exponential function $f(x) = a^x$ for some $a \in \mathbb{R}_{>0} \backslash \{1\}$ is an injective function, or one-to-one. It therefore has an inverse function, which is given by $f^{-1}(x) = \log_a(x)$. The properties that the logarithm function has are "opposite" to the exponential function. That is, we have

† $\log_a(x) + \log_a(y) = \log_a(x \cdot y)$.

† $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$.

† $\log_a(x^k) = k \cdot \log_a(x)$.

† $\log_a(1) = 0$.

† $\log_a(a) = 1$.

† $\log_a(b) = \frac{1}{\log_b(a)}$.

The most natural choice for the number a, called the base, is Euler's number

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.7183.$$

We often write $\ln(x) := \log_e(x)$.

Example 2.2.1. Simplify the expression

$$2\log_{10}(5) + \log_{10}(2) - \log_{10}(20).$$

Proof. We simply observe that

$$\begin{aligned} 2\log_{10}(5) + \log_{10}(2) - \log_{10}(20) &= \log_{10}(25) + \log_{10}(2) - \log_{10}(20) \\ &= \log_{10}(25 \cdot 2) - \log_{10}(20) \\ &= \log_{10}\left(\frac{50}{20}\right) \\ &= \log_{10}\left(\frac{5}{2}\right). \end{aligned}$$

□

Example 2.2.2. Determine the exact value of $\lambda \in \mathbb{R}$ if $x = 2$ solves the equation

$$\ln(4x - a) = 5.$$

Proof. We observe that

$$\begin{aligned} \ln(8 - a) = 5 \quad &\Longrightarrow \quad 8 - a = e^5 \\ &\Longrightarrow \quad a = 8 - e^5. \end{aligned}$$

□

Example 2.2.3. Prove that $\log_e(e) = 1$.

Proof. Let $\log_e(e) = k$, for some $k \in \mathbb{R}$. Then $e^k = e$. It is then clear that $k = 1$ and therefore $\log_e(e) = 1$.

□

Example 2.2.4. Solve the equation $e^{2x+1} = 5$ for $x \in \mathbb{R}$.

Proof.

$$\begin{aligned} e^{2x+1} = 5 \quad &\Longrightarrow \quad \log_e(e^{2x+1}) = \log_e(5) \\ &\Longrightarrow \quad (2x + 1)\log_e(e) = \log_e(5) \\ &\Longrightarrow \quad 2x + 1 = \log_e(5) \\ &\Longrightarrow \quad 2x = \log_e(5) - 1 \\ &\Longrightarrow \quad x = \frac{1}{2}\left(\log_e(5) - 1\right). \end{aligned}$$

□

Example 2.2.5. Solve the logarithmic equation $\log_e(x) + \log_e(3x+1) = 2$.

Proof. We observe that

$$\begin{aligned}
\log_e(x) + \log_e(3x+1) = 2 &\implies \log_e(x \cdot (3x+1)) = 2 \\
&\implies x(3x+1) = e^2 \\
&\implies 3x^2 + x - e^2 = 0 \\
&\implies x = \frac{-1 \pm \sqrt{1^2 - 4(3)(-e^2)}}{3(2)} \\
&\implies x = \frac{-1 \pm \sqrt{1 + 12e^2}}{6}, \\
&\implies x = \frac{-1 + \sqrt{1 + 12e^2}}{6},
\end{aligned}$$

where the last implication is realised since we may only take positive values for x in the domain of the logarithm. □

Exercises

Q1. Solve the following exponential equations.

a. $$e^{3x-5} = 10.$$

b. $$e^{2x-7} = 1.$$

c. $$2e^x = \frac{1}{3}.$$

d. $$e^x \cdot e^{2x+4} = 2.$$

e. $$e^{x-6} = \frac{3}{e^{x+2}}.$$

f. $$e^{4x-5} = \frac{e^{x-1}}{6e^{4-x}}.$$

Q2. Solve the following logarithmic equations for $x \in \mathbb{R}$.

a. $\log_e(x) + \log_e(3) - \log_e(4) = \log_e(x+1)$.

b. $\log_e(2x) + \log_e(4) = 2\log_e(2)$.

c. $\log_e(7-x) + 4\log_e(1) = 2\log_e(x)$.

Q3. Solve the equation

$$\ln(x^2 - 2x + 8) = 2\ln(x),$$

for $x \in \mathbb{R}$.

Q4. Prove that

$$\log_a(b) = \frac{1}{\log_b(a)}.$$

Q5. Determine the value(s) of $k \in \mathbb{R}$ such that the following equation has only one solution for $x \in \mathbb{R}$,

$$\log_e(x) - 3k\log_x(e) + 2 = 0.$$

Q6. Determine the value(s) of $k \in \mathbb{R}$ such that the following equation has no solutions for $x \in \mathbb{R}$,

$$\log_e(x) - \log_x(e) + k = 3.$$

Q7. Let $\mu > 0$ be a fixed real number. Solve the equation

$$e^{\mu x} = 3 + \frac{1}{e^{\mu x}}.$$

Q8. Determine the value(s) of $k \in \mathbb{R}$ such that the equation

$$\log_4(x) - \frac{1}{3}\log_x(4) = 5$$

has one solution.

Q9. Determine the value(s) of $k \in \mathbb{R}$ such that the equation

$$\frac{1}{\sqrt{3}}\log_2(x+3) + \frac{5}{\log_{x+3}(2)} = \frac{\sqrt{3}}{5}$$

has two solutions.

Q10. a. Solve $\log_3(6-x) - \log_3(4-x) = 2$ for x, where $x < 4$.

b. Solve $3e^x = 5 + 8e^{-x}$ for $x \in \mathbb{R}$.

Q11. Determine the value of $k \in \mathbb{R}$ such that

$$4e^x - ke^{-x} = 5,$$

has a unique solution.

Q12. Solve the following equation for $x \in \mathbb{R}$,

$$\log_x(3) + \log_3(x) = 2.$$

Q13. Solve the following equation for $x \in \mathbb{R}$,

$$\log_e(x) + \log_e(x-3) = 4.$$

Q14. Determine the value of $k \in \mathbb{R}$ such that the equation

$$\log_4(x) - k\log_x(4) = 3$$

has two solutions.

Q15. Solve the following equation for $x \in \mathbb{R}$,

$$3^x - 3^{2-x} = 5.$$

Q16. (Dr. Lloyd Gunatilake). Solve the following equation

$$2\log_p 8 - \log_p 4 = 2.$$

Q17. (Dr. Lloyd Gunatilake). Solve the following equation for $x \in \mathbb{R}$,

$$5^{2x^2} - 5^{(x^2+1)} + 6 = 0.$$

Q18. (Dr. Lloyd Gunatilake). Show that

$$\log_a(N)\log_b(N) + \log_b(N)\log_c(N) + \log_c(N)\log_a(N) = \frac{\log_a(N)\log_b(N)\log_c(N)}{\log_{abc}(N)}.$$

Q19. Let f be the function defined by $f(x) = \log_e(x-3) + 1$ and g be the function defined by $g(x) = 3\sqrt{x+4}$.

a. State the domain and range of f and g.

b. Determine the maximal domain of g such that $f(g(x))$ is well-defined.

c. Determine the expression for $f(g(x))$.

d. State the domain and range of $f(g(x))$.

Q20. Let f be the function defined by $f(x) = e^{4x+1}$ and g be the function defined by

$g(x) = \frac{1}{3x-1} + 2$.

a. State the domain and range of f and g.

b. Determine the maximal domain of g such that $f(g(x))$ is well-defined.

c. Determine the expression for $f(g(x))$.

d. State the domain and range of $f(g(x))$.

Q21. Let f be the function defined by $f(x) = x^2 + 1$ and g be the function defined by

$$g(x) = \frac{1}{\log_e(\sqrt{x}) + 1}.$$

a. State the domain and range of f and g.

b. Determine the maximal domain of g such that $f(g(x))$ is well-defined.

c. Determine the expression for $f(g(x))$.

d. State the domain and range of $f(g(x))$.

Q22. Let f be the function defined by $f(x) = e^{-2x+1}$ and g be the function defined by $g(x) = \log_e |x + 2| + \frac{1}{3}$.

a. State the domain and range of f and g.

b. Determine the maximal domain of g such that $f(g(x))$ is well-defined.

c. Determine the expression for $f(g(x))$.

d. State the domain and range of $f(g(x))$.

Q23. Determine the domain and range of the function f defined by

$$f(x) := \frac{1}{3}\log_e\left(\frac{1}{\sqrt{x^2+1}}\right).$$

Q24. Determine the domain and range of the function f defined by

$$f(x) := e^{-x} + e^{x} + \log_e\left(\frac{1}{x}\right).$$

Q25. Determine the domain and range of the function f defined by

$$f(x) := \log_e \left|x^2 - 5x + 6\right| + 2.$$

Q26. Determine the domain and range of the function f defined by

$$f(x) = e^{-x}\log_e(x^2 - 9) + \frac{1}{e^x - 4}.$$

Q27. Determine the value of $k \in \mathbb{R}$ such that the equation

$$\log_e(x) + \log_x(e) = 2k,$$

has a unique solution.

Q28. Determine the value of $k \in \mathbb{R}$ such that

$$3\log_e(x) + \frac{2}{\log_e(x)} + \frac{1}{k} = 4$$

has at least two solutions.

Q29. Determine the value of $k \in \mathbb{R}$ such that

$$2 + \log_e(x) = 4\log_x(e) - 5k$$

has no solutions.

Q30. Determine the value of $k \in \mathbb{R}$ such that

$$ke^x + 2e^{-x} = 4k$$

has two solutions.

Q31. Determine the value of $k \in \mathbb{R}$ such that

$$e^{-x} + \frac{3}{k}e^x + 1 = ke^{-x}$$

has one solution.

Q32. Determine the value of $\lambda \in \mathbb{R}$ such that

$$\frac{4}{\log_e(x)} + 3\lambda\log_e(x) = 1$$

has no solutions.

Q33. Determine the value of $\mu \in \mathbb{R}$ such that

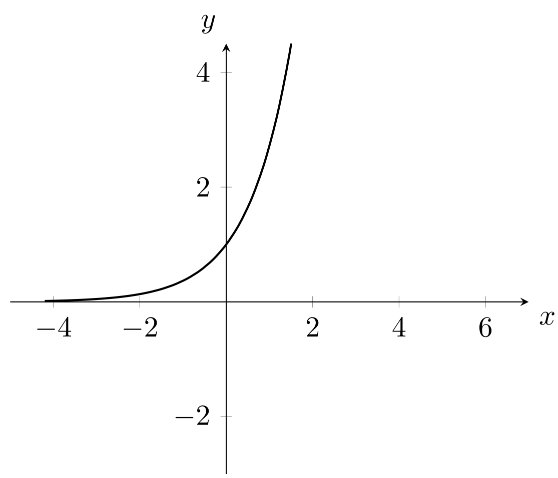
$$2\log_e(\sqrt{x}) + 3\lambda\log_{\sqrt{x}}(e) = \frac{1}{\lambda}$$

has one solution.

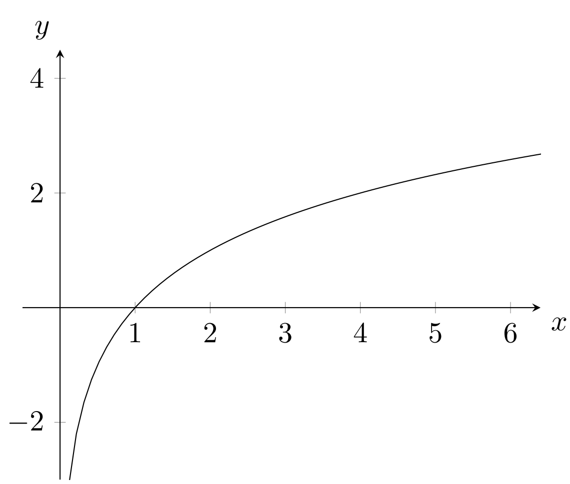
1.3 Sketching the Exponential and Logarithm.

In this section we discuss the methods involved in the graphing of the exponential and logarithm functions.

The graph associated to the curve $f(x) = e^x$ is given by



The graph associated to the curve $f(x) = \log_e(x)$ is given by



Example 2.3.1. State the transformations necessary to map $f(x) = e^x$ to $\widetilde{f}(x) = \frac{1}{3}e^{4-x} + 1$, and hence, sketch the graph of $\widetilde{f}(x)$.

Proof. In order the transformations are given by

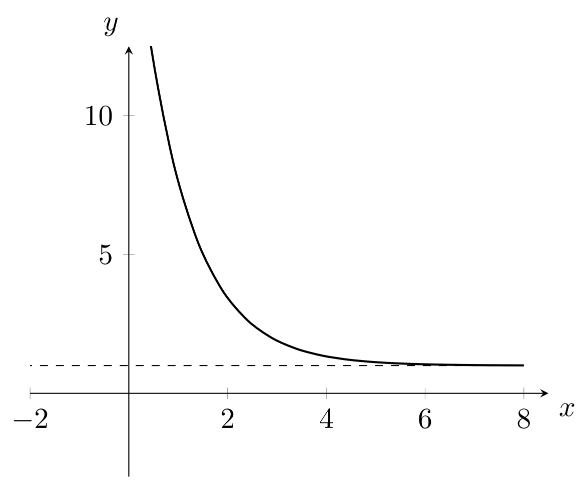
† Dilate by factor $\frac{1}{3}$ from the x-axis.

† Reflect across the y-axis.

† Translate by 4 units in the positive x-direction.

† Translate by 1 unit in the positive y-direction.

The graph is given by



□

Example 2.3.2. Sate the transformations necessary to map $f(x) = \log_e(x)$ to $\widetilde{f}(x) = -2\log_e\left(\frac{1}{3}x - 2\right)$.

Proof. In order the transformations are given by

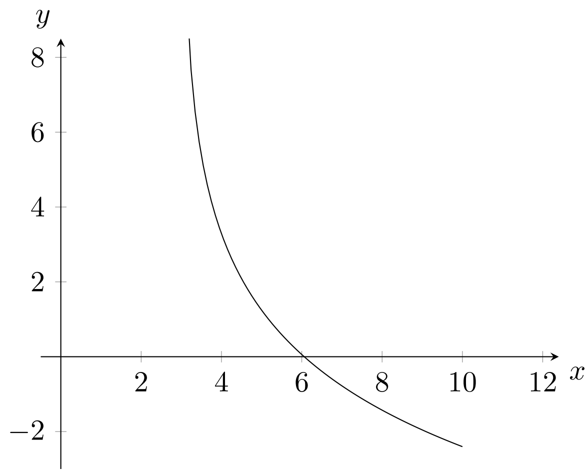
† Dilate by factor 2 from the x-axis.

† Dilate by factor 3 from the y-axis.

† Reflect across the x-axis.

† Translate by 6 units in the positive x-direction.

The graph is given by



□

Exercises

Q1. State the transformations necessary to map $f(x) = e^x$ to the following curves.

a. $\widetilde{f}(x) = 2e^{x+1}$.

b. $\widetilde{f}(x) = 7 - e^x$.

c. $\widetilde{f}(x) = \frac{1}{2} + 4e^{3-x} - 2$.

d. $\widetilde{f}(x) = \frac{1}{3} + e^{9x}$.

e. $\widetilde{f}(x) = 1 + e^{-x}$.

f. $\widetilde{f}(x) = 2 - e^{3(x-1)}$.

Q2. Sketch the following curves, stating all relevant features.

a. $f(x) = 2e^{x+1}$.

b. $f(x) = 7 - e^x$.

c. $f(x) = \frac{1}{2} + 4e^{3-x} - 2$.

d. $f(x) = \frac{1}{3} + e^{9x}$.

e. $f(x) = 1 + e^{-x}$.

f. $f(x) = 2 - e^{3(x-1)}$.

Q3. State the transformations necessary to map $f(x) = \log_e(x)$ to the following curves.

a. $\widetilde{f}(x) = 2\log_e(x-1) + 2$.

b. $\widetilde{f}(x) = \frac{1}{3}\log_e(2-x) - 4$.

c. $\widetilde{f}(x) = 3\log_e(2x+1)$.

d. $\widetilde{f}(x) = 1 - \log_e(-x)$.

e. $\widetilde{f}(x) = \frac{1}{3} - \log_(3x + 3)$.

f. $\widetilde{f}(x) = 2\log_e(\sqrt{3}x - 4) + 1$.

Q4. Sketch the following curves, stating all relevant features.

a. $f(x) = 2\log_e(x-1)+2$.

b. $f(x) = \frac{1}{3}\log_e(2-x)-4$.

c. $f(x) = 3\log_e(2x+1)$.

d. $f(x) = 1-\log_e(-x)$.

e. $f(x) = \frac{1}{3}-\log_(3x+3)$.

f. $f(x) = 2\log_e(\sqrt{3}x-4)+1$.

Q5. State the domain and range of the following functions.

a. $f(x) = e^{4-x}$.

b. $f(x) = \log_e(x-3)+1$.

c. $f(x) = e^{2x-1}+3$.

d. $f(x) = \sqrt{2}\log_e(x+12)$.

Q5. State the domain and range of the following functions.

a. $f(x) = \log_e(\sqrt{x^2-5x+6})$.

b. $f(x) = e^{-\frac{1}{x}}$.

Q6. Let $f(x) = 2e^{x+3}-5$.

a. Determine the inverse function $f^{-1}(x)$.

b. State the domain and range of $f(x)$.

c. State the domain and range of $f^{-1}(x)$.

d. Sketch f and f^{-1} on the same pair of axes.

Q7. Let $f(x) = \frac{1}{5}\log_e(1-x)+3$.

a. Determine the inverse function $f^{-1}(x)$.

b. State the domain and range of $f(x)$.

c. State the domain and range of $f^{-1}(x)$.

d. Sketch f and f^{-1} on the same pair of axes.

Q8. Let $f(x) = \frac{1}{5}e^{2x-6}+2$.

a. Determine the inverse function $f^{-1}(x)$.

b. State the domain and range of $f(x)$.

c. State the domain and range of $f^{-1}(x)$.

d. Sketch f and f^{-1} on the same pair of axes.

Q9. Let $f(x) = 1-2\log_e(3-7x)$.

a. Determine the inverse function $f^{-1}(x)$.

b. State the domain and range of $f(x)$.

c. State the domain and range of $f^{-1}(x)$.

d. Sketch f and f^{-1} on the same pair of axes.

Q10. Let f be the function defined by

$$f(x) = \log_e |x| .$$

a. State the domain and range of f.

b. Sketch the graph of $f(x)$.

c. On what domain(s) does f have an inverse function f^{-1}?

Q11. Sketch the curve $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := e^{|x|}.$$

Q12. Your friend Jimmy seems to have no ability to keep his room clean and so over a period of weeks, a bacterial culture begins to grow within the carpet of his room. On day one of you noticing this, there are 2000 bacteria in his carpet and over the weeks, you notice that the number of bacteria triple every day.

a. Determine the equation that models the growth of the bacteria.

b. How many bacteria are in Jimmy's carpet after 2 weeks?

c. You make the decision that Jimmy needs to know about this when the number of bacteria reaches 100,000. How many days does it take before you decide to tell Jimmy?

Q13. Radium (Ra) has an atomic number of 88 and is an alkaline earth metal. There are currently 33 known isotopes for Radium, mass numbers of which range from 202 to 234; none of these isotopes are stable. ^{223}Ra has a half life of 11.4 days.

a. Determine the half-life of ^{223}Ra in hours.

b. Detemine the value of the decay constant λ, approximate to 3 decimal places.

c. Suppose we initially have 1kg of ^{223}Ra. How many kilograms of ^{223}Ra will we have after 1 day?

d. How long will it take for the original 1kg of ^{223}Ra to decay to 1g?

Q14. In the first few weeks after being born, it is known that the weight of a newborn is given by an expression of the form

$$W(t) = W_0 e^{kt},$$

where $W_0, k \in \mathbb{R}$ and $0 \leq t \leq 10$. Suppose that your newborn cousin Barry weighs 4kgs at birth and after 3 weeks, weighs 4.8kg. How much will Barry weigh after 7 weeks?

Q15. In number of students in the Student Representative Council (SRC) affected by the ebola virus is modelled by the curve

$$\mathscr{I}(t) = \frac{L}{1 + Me^{-kt}},$$

where t is measured in months and $L, M, k \in \mathbb{R}$. In the first month, 3 students were affected, but after another 4 months, the number of infected students grew to 24. If the virus was first introduced by Edmund into the SRC, determine the values of L, M and k up to 3 decimal places.

Q16. Let $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ be the function defined by

$$f(x) := \frac{1}{2} \log_e |x - 3| - 4.$$

Sketch the graph of f, stating all relevant features.

Q17. Let f be the function defined by

$$f(x) := \log_e |2x + 5| + 3.$$

Sketch the graph of f, stating all relevant features.

Q18. Let f be the function defined by

$$f(x) := \frac{2}{5} \log_e |4x + 7| - 2.$$

Sketch the graph of f, stating all relevant features.

Q19. Let f be the function defined by

$$f(x) := e^{|x|}.$$

Sketch the graph of f, stating all relevant features.

Q20. Let f be the function defined by

$$f(x) = \left|e^x - 4\right|.$$

Sketch the graph of f, stating all relevant features.

Q21. Let f be the function defined by

$$f(x) := \left|1 - \log_e(x+2)\right|.$$

Sketch the graph of f, stating all relevant features.

Q22. Let f be the function defined by

$$f(x) = \left|\log_e |x|\right|.$$

Q23. Let f be the function defined by

$$f(x) = \left|e^{1-4x} - 5\right| + 2.$$

Q24. Let f be the function defined by $f(x) := \log_e(x)$ and let g be the function defined by

$$g(x) = \frac{1}{3}\log_e(1-x) + 2.$$

Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q25. Let f be the function defined by $f(x) := \log_e(x)$ and let g be the function defined by

$$g(x) = 3 - \frac{4}{5}\log_e(2x-5).$$

Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q26. Let f be the function defined by $f(x) := \log_e(x)$ and let g be the function defined by

$$g(x) := 3 + 2\log_e(x+10).$$

Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q27. Let f be the function defined by $f(x) := e^x$ and let g be the function defined by

$$g(x) := \frac{1}{3}e^{4x+1} - 7.$$

Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q28. Let f be the function defined by $f(x) := e^x$ and let g be the function defined by

$$g(x) = \frac{1}{4e^{x-5}} + 2.$$

Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q29. Let f be the function defined by

$$f(x) := \frac{3}{7}\log_e(x+1) - 2$$

and let g be the function defined by

$$g(x) := \frac{3}{5}\log_e(x+2) + 4.$$

Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q30. Let f be the function defined by

$$f(x) = 3 - \log_e(2x+5)$$

and let g be the function defined by

$$g(x) := \frac{5}{7}\log_e(x+8) - 1.$$

Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q31. Let f be the function defined by

$$f(x) = \frac{1}{5}\log_e(1-x) + 10$$

and let g be the function defined by $g(x) := \log_e(x)$. Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q32. Let f be the function defined by

$$f(x) := \frac{4}{5}e^{4-x} + 1$$

and g be the function defined by

$$g(x) = 1 - e^{-x+10}.$$

Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q33. Let f be the function defined by

$$f(x) := \frac{1}{2}e^{2x-7} + 13$$

and let g be the function defined by $g(x) := e^x$. Describe the transformations required to map $f(x)$ to $g(x)$ and write out the corresponding transformation matrix.

Q34. State the domain of the function

$$f(x) := \log_e\left(\frac{e^x}{\sqrt{x^2 - 5x + 6}}\right).$$

1.4 Review Exercises

Q1. Solve the equation

$$2\log_e(x) + \frac{1}{3} = \frac{1}{2}\log_e(x) + 2,$$

for $x \in \mathbb{R}_{>0}$.

Q2. Solve the equation

$$3^x - 5 + \frac{6}{3^x} = 0,$$

for $x \in \mathbb{R}$.

Q3. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \log_e(x)$. Let $\widetilde{f} : (-\infty, 1) \to \mathbb{R}$ be defined by $\widetilde{f}(x) = \frac{1}{3}\log_e(1 - x) + 2$.

a. State the transformations necessary to map $f(x)$ to $\widetilde{f}(x)$.

b. Sketch the graphs of $f(x)$ and $\widetilde{f}(x)$ on the same pair of axes, stating all relevant features.

Q4. Let f be the function defined by $f(x) := \frac{1}{3}e^{4-3x}$, and let g be the function defined by $g(x) := \frac{3x+1}{7-x}$.

a. State the domain and range of $f(x)$ and $g(x)$.

b. Determine the maximal domain of $f(x)$ such that $g(f(x))$.

c. State the rule for $g(f(x))$.

d. Determine the domain and range of $g(f(x))$.

Q5. Determine the value of $k \in \mathbb{R}$ such that the equation

$$5\log_e(x) + 3k = 2k\log_x(e)$$

has no solutions.

Q6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \frac{1}{4}e^{2x+1} - 8.$$

Determine the rule for the inverse function $f^{-1}(x)$.

Q7. Solve the equation

$$\log_2(x + 1) - \log_2(x - 3) = \log_2(x + 5),$$

for $x \in \mathbb{R}$.

Q8. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 1 - 2e^{4+x}$. Determine the equation $\widetilde{f}(x)$ that is obtained from applying the following transformations to f.

1. Dilate by factor 2 from the x-axis.
2. Translate by 3 units down.
3. Reflect across the y-axis.

Q9. State the domain of the function

$$f(x) = \frac{1}{2}\log_e\left(1 - \frac{1}{\sqrt{x^2 + 3x - 4}}\right).$$

Q10. Let f be the function defined by

$$f(x) = 3 - \log_e(1 - x).$$

Sketch the graph of f on a suitable domain, stating all relevant features.

Q11. Determine the value(s) of $k \in \mathbb{R}$ such that the equation

$$\log_e(1 - x) + k\log_{1-x}(e) = 4$$

has no solutions.

Q12. Let $f(x) = 1 - \frac{1}{2}e^{4-x}$. Determine the inverse function $f^{-1}(x)$.

Q13. Graph the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 1 - e^{|x|}.$$

Q14. Prove that

$$\log_a(b) = \frac{1}{\log_b(a)}.$$

Q15. Solve the equation

$$e^x = 7 - 12e^{-x},$$

for $x \in \mathbb{R}$.

Q16. State the transformations necessary to maps $f(x) = \log_e(1 - x)$ to

$$\widetilde{f}(x) = 4 - \frac{1}{2}\log_e(4 - 2x).$$

Q17. Let f be the function defined by

$$f(x) = \frac{1}{3} - \log_e |x| .$$

Sketch the graph of f on a suitable domain, stating all relevant features.

Q18. State the domain of

$$f(x) := 4 + \log_4 \left(1 - \frac{1}{\sqrt{x^2 + 2x + 1}} \right) .$$

Q19. Sketch the inverse function of

$$f(x) = 4 - e^{4x+1},$$

stating all relevant features.

Q20. Prove that

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}.$$

Q21. Determine the value(s) of $k \in \mathbb{R}$ such that

$$\log_{\sqrt{x}-5x+1}(e) + k \log_e(\sqrt{x} - 5x + 1) = 4k$$

has a unique solution.

Q22. Determine the domain of the function

$$f(x) = \log_e \left(\log_e(\sqrt{x}) \right) .$$

Q23. Solve the equation

$$5^{-2x^3} - 5^{1-x^3} = -5,$$

for $x \in \mathbb{R}$.

Q24. Determine the inverse function of

$$f(x) = \frac{1}{2} + 2 \log_e(1 + 4x).$$

Q25. Sketch the graph of the function

$$f(x) = 3 - e^{2-x}.$$

Q26. Determine the transformations required to map $f(x) = \log_e(x)$ to

$$\widetilde{f}(x) = \frac{1}{3} - 2\log_e(5 + x).$$

Q27. Solve the equation

$$\log_e(x) + 3 = 4\log_x(e).$$

Q28. Let $f(x) = 1 - 2\log_e(x)$ and $g(x) = \log_e(3 - x) + 1$. Determine the point of intersection between f and g.

Q29. Determine the value(s) of $k \in \mathbb{R}$ such that

$$e^{-x} + ke^{x} = 4 - k$$

has two solutions.

Q30. Solve the equation

$$1 - \left|e^{2-x} + 4\right| = 0,$$

for $x \in \mathbb{R}$.

Q31. Determine the inverse function of

$$f(x) = 4 - \sqrt{3}e^{4-x}.$$