Curvature and Moduli

Some Intimations and Propaganda

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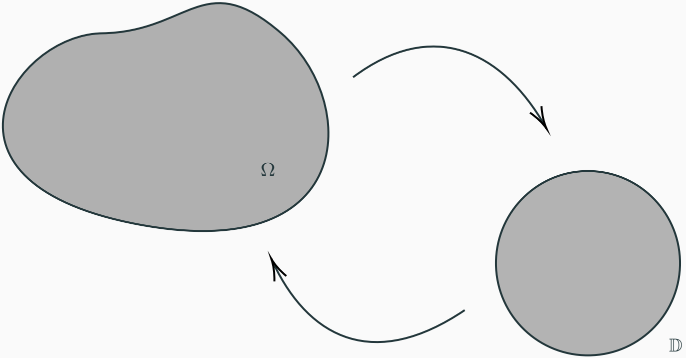
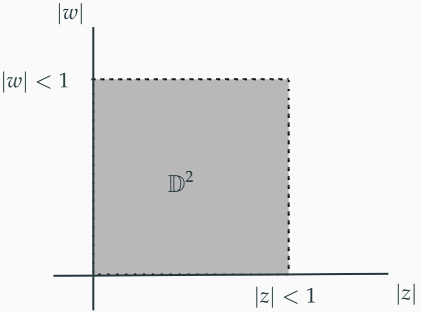
Riemann Mapping Theorem

Theorem. (Riemann, Koebe). A simply connected domain $\Omega \subsetneq \mathbb{C}$ is biholomorphic to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

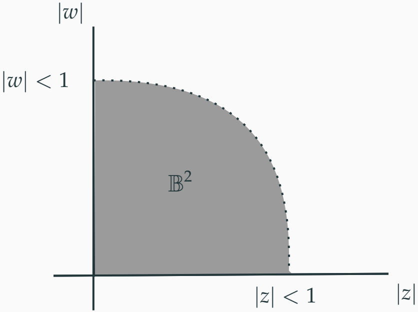


The Birth of Several Complex Variables

Theorem. (Poincaré). The ball $\mathbb{B}^2 := \{|z|^2 + |w|^2 < 1\}$ is not biholomorphic to the bidisk $\mathbb{D}^2 := \{|z| < 1, |w| < 1\}$.



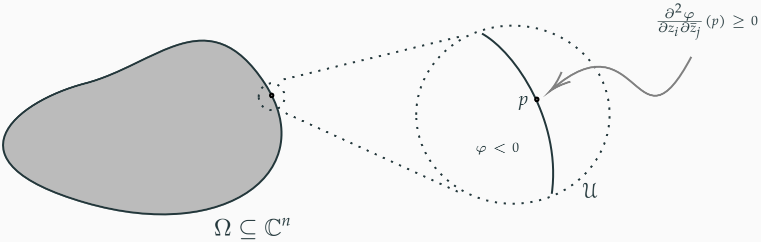
Bidisk



The Ball

Pseudoconvexity

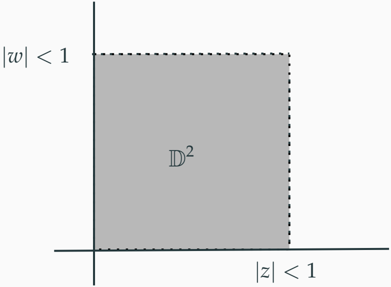
Definition. A bounded domain $\Omega \subset \mathbb{C}^n$ is (strongly) pseudoconvex if for all $p \in \partial\Omega$, there is a smooth local defining function φ such that complex Hessian $\partial\bar{\partial}\varphi = \left(\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right)$ is (strictly) positive definite at p.



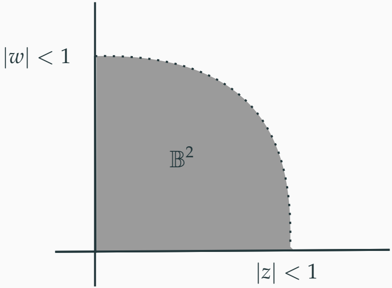
Pseudoconvexity of the Ball $\mathbb{B}^2$ and the Bidisk $\mathbb{D}^2$.

The bidisk $\mathbb{D}^2$ is pseudoconvex while the ball $\mathbb{B}^2$ is strongly pseudoconvex.

Pseudoconvexity and strong pseudoconvexity are preserved under biholomorphism. Hence, $\mathbb{D}^2$ and $\mathbb{B}^2$ cannot be biholomorphic.



Pseudoconvex

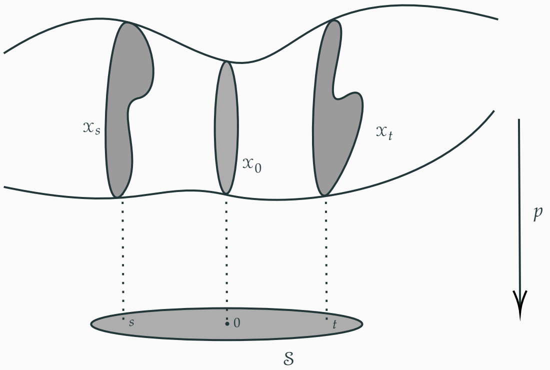


Strongly Pseudoconvex

Disk Fibrations

This discrepancy has consequences on the behavior of disk fibrations:

Definition. A surjective holomorphic submersion $p : \mathcal{X} \to \mathcal{S}$ is said to be a disk fibration if every fiber $\mathcal{X}_s := p^{-1}(s)$, for $s \in \mathcal{S}$, is biholomorphic to a disk.



The Bidisk $\mathbb{D}^2$ as a Disk Fibration

Reminder. We say that a disk fibration $p : \mathcal{X} \to \mathcal{S}$ is locally (holomorphically) trivial if for each point $s \in \mathcal{S}$, there is an open neighborhood $\mathcal{U} \ni s$ such that

$$p^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{D}.$$

Of course, if $\mathcal{X} = \mathbb{D}^2$, for any point $s \in \mathbb{D}$, we can take $\mathcal{U} = \mathbb{D}$. Hence, for the bidisk $\mathbb{D}^2$, the disk fibration $p : \mathbb{D}^2 \to \mathbb{D}$ is holomorphically trivial.

The Ball $\mathbb{B}^2$ as a Disk Fibration

On the other hand, the disk fibration $p : \mathbb{B}^2 \to \mathbb{D}$ cannot be holomorphically trivial:

An old theorem of Royden tells us that a disk fibration is locally holomorphically trivial if and only if it is holomorphically trivial. Hence, if $p : \mathbb{B}^2 \to \mathbb{D}$ is locally trivial, then $\mathbb{B}^2$ would be biholomorphic to $\mathbb{D}^2$.

Hence, from the viewpoint of moduli and deformation theory, the bidisk $\mathbb{D}^2$ and the ball $\mathbb{B}^2$ behave very differently.

Disk Fibrations and Curvature

Understanding the behavior of complex manifolds in families can be difficult, and we would like to have a robust mechanism for measuring the existence or non-existence of holomorphic variation in the fibers.

Question. Can the behavior of the disk fibrations $p : \mathcal{X} \to \mathcal{S}$ be detected by looking at the curvature of metrics which reside on $\mathcal{X}$?

Definition. We say that a complete Riemannian manifold (M, g) satisfies the unique geodesic property if for any $p, q \in M$, there is a unique geodesic connecting p and q that minimizes the length in its homotopy class.

Key Lemma. Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature. Then (M, g) supports the unique geodesic property.

Corollary. (Cartan–Hadamard). Let (M, g) be a complete Riemannian manifold supporting the unique geodesic property. Then the universal cover $\widetilde{M} \simeq_{\text{diffeo.}} \mathbb{R}^n$.

Proof. $\pi_1(\widetilde{M}) = 0 \implies$ only one homotopy class.
$\implies$ unique geodesic connecting any two points of $\widetilde{M}$.
$\implies$ exponential map $\exp_p : T_p\widetilde{M} \to \widetilde{M}$ is bijective.
$\implies$ exponential map is a diffeomorphism.

Theorem. (Priessman). Let (M, g) be a compact Riemannian manifold with $\mathrm{Sec}_g < 0$. Then every abelian subgroup of $\pi_1(M)$ is infinite cyclic.

Proof. Let $\alpha, \beta \in \pi_1(M, p)$ be two commuting loops.
Homotopy between $\alpha\beta$ and $\beta\alpha \implies \exists\, f : \mathbb{T}^2 \to M$ (continuous).
$\mathrm{Sec}_g < 0 + \mathrm{ES\ Thm} \implies f \simeq_{\text{homotopic}} f^{\mathrm{H}} : \mathbb{T}^2 \to M$ (harmonic).
$\mathrm{Sec}_g < 0 \implies f^{\mathrm{H}}(\mathbb{T}^2) \subset \gamma$ (closed geodesic).
$\implies$ loops in $\pi_1(M)$ given by α and β through the homotopy from f to f^{H} are multiples of γ.
$\implies$ contained in a cyclic subgroup of $\pi_1(M)$.
Unique geodesic property $\implies$ cyclic group is infinite.
$\implies \{\alpha, \beta\} \subset \pi_1(M)$ is infinite cyclic.

Corollary. Compact Riemannian manifolds (M, g) with $\mathrm{Sec}_g < 0$ cannot be homeomorphic to products.

Proof. Suppose $M \simeq X \times Y$.
Cartan–Hadamard $\implies \pi_k(M) = 0,\ k > 1$ (i.e., M is aspherical).
$\implies X, Y$ are aspherical.
M compact $\implies X, Y$ compact $\implies \pi_1(X) \neq 0$ and $\pi_1(Y) \neq 0$.
$\implies \exists \gamma_X \neq 0 \in \pi_1(X), \gamma_Y \neq 0 \in \pi_1(Y)$.
$\implies \{\gamma_X\} \simeq \mathbb{Z} \subset \pi_1(X),\ \{\gamma_Y\} \simeq \mathbb{Z} \subset \pi_1(Y)$.
$\implies \{\gamma_X, \gamma_Y\} \simeq \mathbb{Z} \oplus \mathbb{Z}$.
Violates Priessman's theorem.

Negative Curvature and Products

Without compactness, negative sectional curvature is not obstructed on products:

Theorem. (Anderson). Let $f : \mathcal{E} \to \mathcal{B}$ be a smooth vector bundle over a compact Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$ with $\mathrm{Sec}_{g_{\mathcal{B}}} < 0$. Then $\mathcal{E}$ admits a complete Riemannian metric $g_{\mathcal{E}}$ with

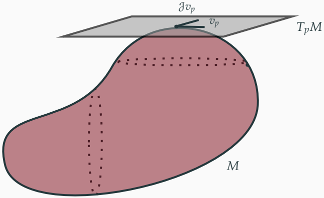
$$-a \;\leq\; \mathrm{Sec}_{g_{\mathcal{E}}} \;\leq\; -1.$$

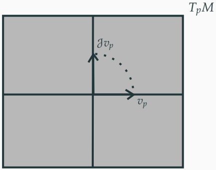
Theorem. (Bishop–O'Neill). There is a complete metric of constant negative curvature on $\mathbb{R} \times \mathcal{F}$, where $\mathcal{F}$ is any compact Riemannian manifold with a flat metric.

Complex Structures

Definition. An almost complex structure $\mathcal{J}$ on a smooth manifold M is an endomorphism

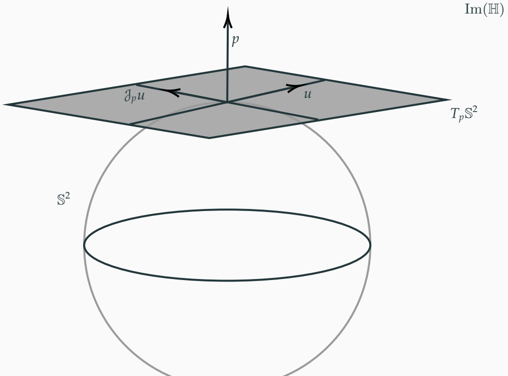
$$\mathcal{J} : TM \to TM, \qquad \mathcal{J}^2 = -\mathrm{id}.$$





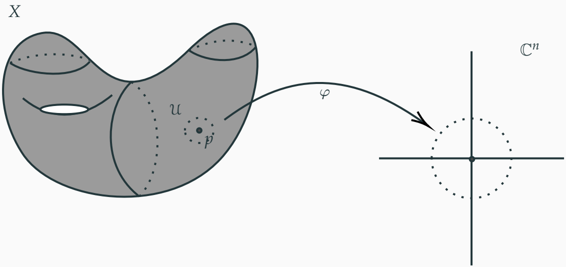
An Almost Complex Structure on $\mathbb{S}^2$

Identify $\mathbb{S}^2 \subset \mathbb{R}^3$ with the space of unit imaginary quaternions $\mathrm{Im}(\mathbb{H}^3) \simeq \mathbb{R}^3$. For each point $p \in \mathbb{S}^2$, we get a map $\mathcal{J}_p : T_p\mathbb{S}^2 \to T_p\mathbb{S}^2$ satisfying $\mathcal{J}_p^2 = -\mathrm{id}_{T_p\mathbb{S}^2}$, given by $\mathcal{J}_p(v) := p \times v$.



In general, an almost complex structure $\mathcal{J} \in \mathrm{End}(TX)$ is not sufficient to yield local holomorphic coordinates. There is an obvious obstruction: Suppose X is a complex manifold with holomorphic coordinates $(z_1, ..., z_n)$ centered at a point $p \in X$. The tangent space to X at the point p is the complex vector space:

$$T_pX \ = \ \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n} \right\}.$$



Obstruction to Local Holomorphic Coordinates

Let M be a smooth manifold with almost complex structure $\mathcal{J}$. The condition $\mathcal{J}^2 = -\mathrm{id}$ gives an eigenspace splitting

$$T_p^{\mathbb{C}}M \;\simeq\; T_p^{1,0}M \oplus T_p^{0,1}M,$$

corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

If $(x_1, \dots, x_{2n})$ are smooth coordinates on M, then $T_p^{1,0}M$ is spanned by

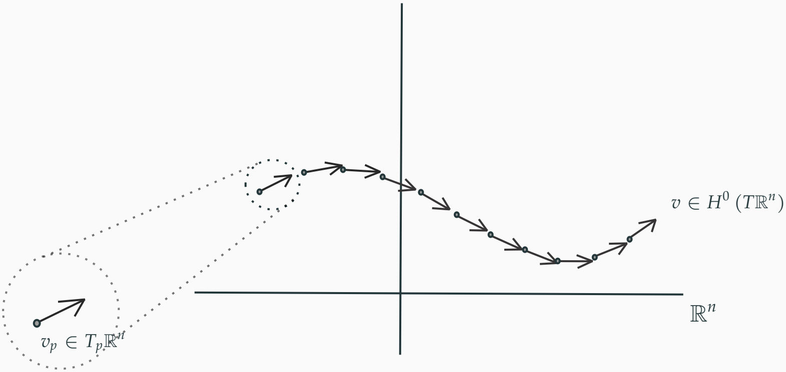
$$\frac{\partial}{\partial z_i} \;:=\; \frac{\partial}{\partial x_i} - \sqrt{-1}\mathcal{J}\frac{\partial}{\partial x_i},$$

and $T_p^{0,1}M$ is spanned by

$$\frac{\partial}{\partial \overline{z}_i} \;:=\; \frac{\partial}{\partial x_i} + \sqrt{-1}\mathcal{J}\frac{\partial}{\partial x_i}.$$

Vector Fields and Integral Curves

We have seen this before in the context of vector fields and integral curves:



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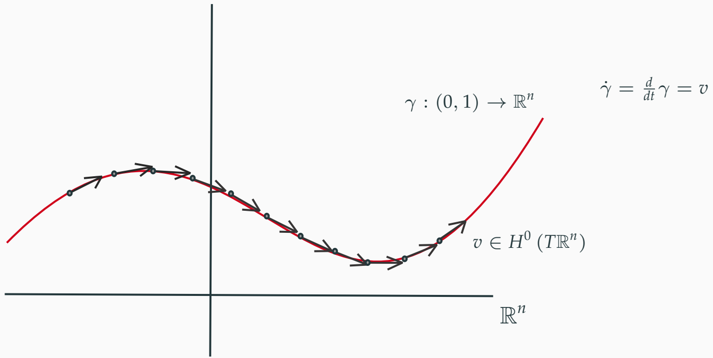
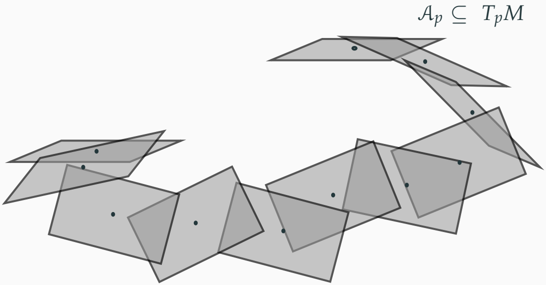


Integrability

The integrability condition on the complex structure is merely a higher-dimensional version of this:



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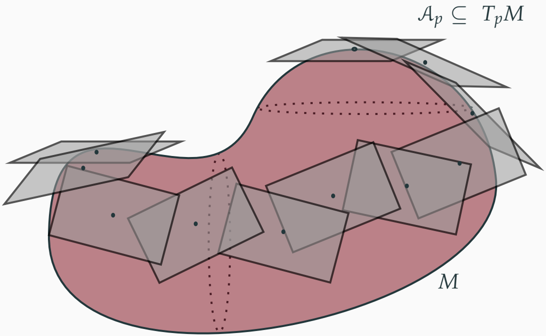


The Newlander–Nirenberg Theorem

The Frobenius theorem tells us that $T^{1,0}M$ is an integrable subbundle if and only if it is closed under Lie bracket:

$$[u, v] \subseteq T^{1,0}M, \qquad \forall u, v \in T^{1,0}M.$$

This manifests as the vanishing of the Nijenhuis tensor:

$$\mathcal{N}^{\mathcal{J}}(u_0, v_0) \quad := \quad [u_0, v_0] + J([Ju_0, v_0] + [u_0, Jv_0]) - [Ju_0, Jv_0].$$

Theorem. (Newlander–Nirenberg). An almost complex structure $\mathcal{J}$ is integrable if and only if $\mathcal{N}^{\mathcal{J}} \equiv 0$.

A Non-Integrable Almost Complex Structure on $\mathbb{S}^6$

We can repeat the almost complex structure construction on $\mathbb{S}^2$ with $\mathbb{S}^6$ – identify $\mathbb{S}^6$ with the space of unit imaginary octonions $\mathrm{Im}(\mathbb{O})$. This endows $\mathbb{S}^6$ with an almost complex structure.

If one computes the Nijenhuis tensor of this almost complex structure, however, it does not vanish precisely because the octonions are not associative.

Definition. A Riemannian metric g on a complex manifold $(X, \mathcal{J})$ is said to be Hermitian if

$$g(\mathcal{J}u, \mathcal{J}v) \;=\; g(u,v), \qquad\qquad u, v \in TX.$$

We say that a Hermitian metric g is Kähler if the 2–form

$$\omega_g(u,v) := g(\mathcal{J}u, v)$$

is closed.

Examples

Examples of Kähler Manifolds.

† Complex projective space $\mathbb{P}^n$ endowed with the Fubini–Study metric.

  $\rightsquigarrow$ Projective manifolds.

† Euclidean space $\mathbb{C}^n$ endowed with the Euclidean metric.

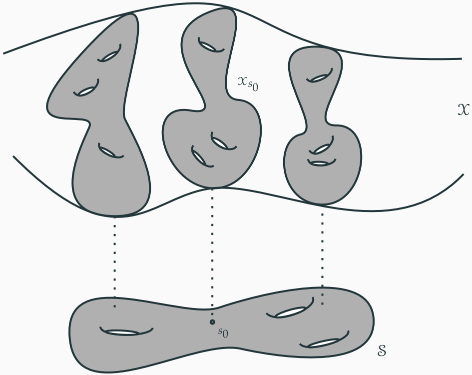
  $\rightsquigarrow$ Stein manifolds (in particular, pseudoconvex domains).

† A compact complex surface is Kähler if and only if the first Betti number is even.

Examples of non-Kähler Manifolds.

$\rightsquigarrow$ Hopf surface $\mathbb{S}^1 \times \mathbb{S}^3$ is not Kähler.

Definition. Let $p : \mathcal{X} \to \mathcal{S}$ be a surjective proper holomorphic submersion onto a hyperbolic curve with hyperbolic fibers. If there fibers are not all biholomorphic, then we say that $p : \mathcal{X} \to \mathcal{S}$ is a Kodaira Fibration Surface.



Sectional Curvature

The Sectional Curvature is a Riemannian invariant, not a complex-analytic invariant.

Let (M, g, J) be Kähler. Compexifying the Riemannian curvature tensor R gives a quadrilinear map R on $T^{\mathbb{C}}M \oplus \overline{T^{\mathbb{C}}M}$ with the only non-trivial components given by

$$R(u, \overline{v}, w, \overline{z}), \qquad u, v, w, z \in T^{\mathbb{C}}M.$$

Hence, the natural Hermitian replacement for the sectional curvature is given by

$$\mathrm{HBC}_{\omega}(u, v) \ := \ R(u, \overline{u}, v, \overline{v}).$$

The Sectional Curvature

Set $u = \frac{1}{\sqrt{2}}\left(u_0 - \sqrt{-1}Ju_0\right)$ and $v = \frac{1}{\sqrt{2}}\left(v_0 - \sqrt{-1}Jv_0\right)$.

The Bianchi identity gives

$$R(u, \overline{u}, v, \overline{v}) \quad = \quad R(v_0, u_0, u_0, v_0) + R(Ju_0, v_0, v_0, Ju_0).$$

In particular, $R(u, \overline{u}, v, \overline{v})$ is a sum of two sectional curvatures, and we therefore call it the holomorphic bisectional curvature.

The Holomorphic Bisectional Curvature

The bisectional curvature is obviously weaker than the sectional curvature, but it is still a very restrictive curvature constraint:

Theorem. (Mori, Siu–Yau). A compact Kähler manifold with HBC > 0 is biholomorphic to $\mathbb{P}^n$.

Theorem. (Mohsen). There are compact simply connected projective manifolds with HBC < 0.

Of course, Mohsen's examples cannot admit metrics with Sec < 0 by the Cartan–Hadamard theorem.

Question. Let $f : \mathcal{E} \to \mathcal{B}$ be a holomorphic vector bundle, where $\mathcal{B}$ is compact and admits a Hermitian metric ω with ${}^c\text{HBC}_\omega < 0$. Does $\mathcal{E}$ admit a complete Hermitian metric with $-a \leq {}^c\text{HBC} \leq -1$, for some constant $a > 1$?

The answer turns out to be false, by a result of F. Zheng:

Theorem. (Zheng). Let $\mathcal{X} := X \times Y$ be a product complex manifold with X compact. Then $\mathcal{X}$ does not admit a Hermitian metric ω with ${}^{c}\mathrm{HBC}_{\omega} \leq -1$.

A Theorem of Paul Yang

Theorem. (Yang). Let $\mathcal{F} \hookrightarrow \mathcal{X} \to \mathcal{B}$ be a holomorphic fiber bundle with $\mathcal{F}$ compact. Then $\mathcal{X}$ does not admit a complete Kähler metric with $\text{HBC}_{\omega} \leq -\kappa_0 < 0$.

The following theorem of Fischer and Grauert shows that holomorphic fiber bundles with compact fiber are trivial in the following sense:

Theorem. (Fischer–Grauert). Let $p : \mathcal{X} \to \mathcal{S}$ be a holomorphic family of compact complex manifolds. The fibers of p are all biholomorphic if and only if p is a holomorphic fiber bundle.

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Corollary. Let $p : \mathcal{X} \to \mathcal{B}$ be a holomorphic family of compact complex manifolds. If $\mathcal{X}$ admits a complete Kähler metric with $\text{HBC}_\omega \leq -\kappa_0 < 0$, there must be non-trivial holomorphic variation in the fibers.

Necessity of the Upper Bound

The bisectional curvature must be bounded away from zero:

Theorem. (Klembeck). There is a complete Kähler metric on $\mathbb{C}^n$ with $\text{HBC}_\omega > 0$.

Seshadri gave a small modification of Klembeck's construction, showing:

Theorem. (Seshadri = Klembeck $+\varepsilon$). There is a complete Kähler metric on $\mathbb{C}^n$ with $\text{HBC}_\omega < 0$.

The Conjectural Picture

Conjecture. Let $f : \mathcal{X} \to \mathcal{S}$ be a holomorphic family of complex manifolds. Suppose $\mathcal{X}$ admits a complete Hermitian metric with HBC $\leq -\kappa_0 < 0$. Then f is not (holomorphically) locally trivial.

Curvature of Kodaira Fibration Surfaces

Theorem. (To–Yeung). Let $p : \mathcal{X} \to \mathcal{S}$ be a Kodaira fibration surface. Then $\mathcal{X}$ admits a Kähler metric with $\text{HBC}_{\omega} < 0$.

A Question Raised by Ngaiming Mok

Question. (Mok). Does the bidisk $\mathbb{D}^2 := \mathbb{D} \times \mathbb{D}$ admit a complete Kähler metric with $\mathrm{HBC}_\omega \leq -\kappa_0 < 0$?

Thanks for listening!