Einstein 4-manifolds with Symmetry

Kyle Broder
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UCSB Geometry Seminar

This talk is based on the following two papers:

The first paper appeared on the arXiv on May 24 2026:

COHOMOGENEITY-ONE EINSTEIN METRICS ON CLOSED 4-MANIFOLDS

KYLE BRODER

Abstract. We classify the Einstein metrics on closed 4-manifolds whose isometry group has dimension at least 3. Consequently, we show that the cohomogeneity-one Einstein metrics on a closed 4-manifold are locally symmetric or homothetic to the Page metric on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2}$.

The second paper appeared on the arXiv on October 13 2026, although the results were all obtained by July–August.

T^2-INVARIANT EINSTEIN METRICS ON CLOSED 4-MANIFOLDS

KYLE BRODER AND RAMIRO A. LAFUENTE

Abstract. We construct T^2-invariant Einstein metrics with positive Ricci curvature on $\sharp_m(S^2 \times S^2)$ for every integer $m > 1$. These provide an infinite family of Einstein metrics on closed 4-manifolds with $\lambda > 0$, which are in addition not Hermitian. We further establish the uniqueness of the T^2-invariant Einstein metrics on S^4, $\mathbb{CP}^2$, $S^2 \times S^2$, and $\mathbb{CP}^2 \sharp\, \mathbb{CP}^2$.

Related works

Oct 6 **Ozuch** – *Positive Einstein metrics on infinitely many four-manifolds.*

Oct 7 **Li–Sun** – *Compact toric Einstein 4-manifolds*

Oct 8 **Chen–Dong** – *Einstein metrics with torus symmetry on S^4 and $\mathbb{R}^4$*

Oct 13 **B.–Lafuente** – *T^2-invariant Einstein metrics on closed 4-manifolds*

Oct 21 **Chu–Stern** – *A variational construction for Einstein metrics on connected sums of $S^2 \times S^2$*

All papers involved significant AI use (and significant hard work).

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$$\mathrm{Ric}(g) = \lambda g, \qquad \lambda \in \mathbb{R}.$$

We will only consider Einstein metrics with $\lambda > 0$.

Classical examples are the round metric on S^4, Fubini–Study metric on $\mathbb{CP}^2$, and the product metric on $S^2 \times S^2$.

Page (1978) discovered a Hermitian (non-Kähler) Einstein metric on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$. This metric has cohomogeneity-one.

Tian–Yau (1987) and Tian (1990) showed there is a Kähler–Einstein metric on $\mathbb{CP}^2 \sharp k\overline{\mathbb{CP}^2}$ for $3 \leq k \leq 8$. When $k = 3$ the metric is T^2-invariant. For $5 \leq k \leq 8$, there are continuous families of KE metrics.

Chen–LeBrun–Weber (2008) constructed a Hermitian (non-Kähler) Einstein metric on $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}$. This metric is T^2-invariant.

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Main theorem

Theorem. (B.–Lafuente, Oct 2026). For every $m \geq 1$, there exists a T^2-invariant Einstein metric on $\sharp_m(S^2 \times S^2)$.

In particular, there are now **infinitely many diffeomorphism types** of closed 4-manifolds known to admit Einstein metrics with $\lambda > 0$. Moreover, for $m > 1$, these metrics **cannot be locally Hermitian**, since this would force $b_2^+ \leq 1$, and $b_2^+(\sharp_m(S^2 \times S^2)) = m$.

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Orlik–Raymond (1970) classified the diffeomorphism-types of closed simply connected 4-manifolds admitting a T^2-action. Together with **Hitchin–Thorpe (1969–1974)**:

Theorem. If a closed simply connected 4-manifold admits an Einstein metric invariant under a T^2-action, then M is diffeomorphic to

$$S^4, \qquad \sharp_m(S^2 \times S^2), \qquad \sharp_p \mathbb{CP}^2 \sharp_q \overline{\mathbb{CP}}^2,$$

where $m \geq 1$, $p, q \geq 0$, $p + q > 0$, and $3|p - q| < 2(2 + p + q)$.

Among these candidates, S^4, $\sharp_r \mathbb{CP}^2$, and $\sharp_r \overline{\mathbb{CP}}^2$ for $1 \leq r \leq 3$ admit a unique T^2-action, while the remaining diffeomorphism types admits countably infinitely many (up to equivalence[1]).

[1] Two actions are equivalent if there exist a diffeomorphism $F : M \to M'$ and $\alpha \in \mathrm{Aut}(\mathsf{T}^2)$ satisfying $F(t \cdot x) = \alpha(t) \cdot F(x)$.

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Comparison with the cohomogeneity-one classification

From the classification results of **Parker (1986)**, **Hoelscher (2007)**, and **Grove–Ziller (2011)**:

Theorem. If a closed simply connected 4-manifold M admits an Einstein metric invariant under a cohomogeneity-one action, then M is diffeomorphic to

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In this case, S^4 admits five actions, $\mathbb{CP}^2$ admits three, while $S^2 \times S^2$ and $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ admit countably infinitely many.

By results of **Berard-Bergery (1982)** and **Derdzinski (1983)**, the classification of cohomogeneity-one Einstein metrics in dimension 4 reduces to the $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$-actions.

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T^2-invariant Einstein

Let (M, g) be a closed simply connected oriented Einstein 4-manifold with an isometric T^2-action.

Write $\pi : M \to \Sigma := M/\mathsf{T}^2$. The orbit space Σ is a closed disk whose interior Σ° parametrizes the free T^2-orbits; the $N = \chi(M)$ vertices correspond to fixed points and each open edge E_i corresponds to circle orbits.

S^4

$\mathbb{CP}^2$

$S^2 \times S^2$

$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$

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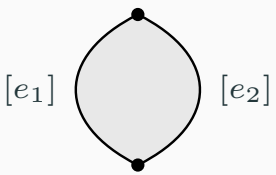
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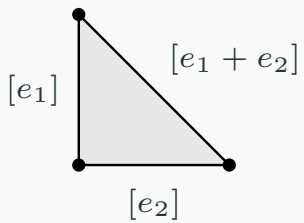
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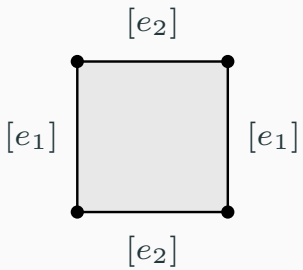
S^4



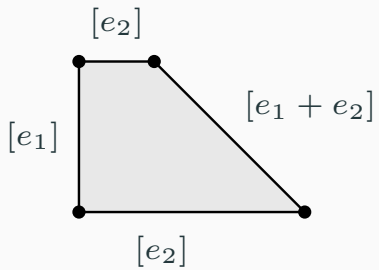
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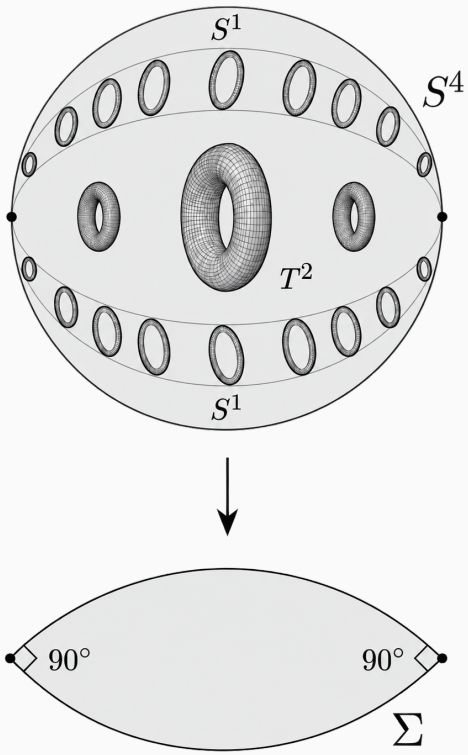


$S^2 \times S^2$



$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$




S^1
S^4
T^2
S^1
90°
90°
Σ

Theorem. (Liu, 2025). An isometric T^2-action on a closed simply connected Einstein 4-manifold is **polar**. In particular, for a basis $K_1, K_2 \in \mathrm{Lie}(\mathsf{T}^2)$, set $G_{ij}(x) := \mathrm{g}(K_i, K_j)$, we have

$$\mathrm{g} \;=\; \pi^*\hat{\mathrm{g}}(x) + G_{ij}(x)\,\mathrm{d}\vartheta^i\mathrm{d}\vartheta^j, \qquad \vartheta^i \in \mathbb{R}/2\pi\mathbb{Z},$$

where $\hat{\mathrm{g}}$ is the quotient metric.

For $x \in \Sigma^\circ$, the area of the orbit $\mathsf{T}^2 = \pi^{-1}(x)$ is $4\pi^2\rho(x)$, where $\rho = \sqrt{\det G}$. Setting $\Phi = \rho^{-1}G$ so that $\det(\Phi) = 1$, we get a **weighted harmonic map** $\Phi : \Sigma^\circ \to \mathbb{H}^2$, i.e., a critical point of

$$\frac{1}{2}\int_{\Sigma^\circ} |\mathrm{d}\Phi|^2\,\rho\,\mathrm{d}v_{\hat{\mathrm{g}}}.$$

With $f = \log\rho$,

$$\mathrm{Ric}(\mathrm{g}) = \lambda\mathrm{g} \iff \begin{cases} \tau_f(\Phi) := \mathrm{tr}_{\hat{\mathrm{g}}}\,\nabla\mathrm{d}\Phi + \mathrm{d}\Phi(\hat{\nabla}f) = 0, \\ \Delta f + |\mathrm{d}f|^2 = -2\lambda, \\ \hat{\nabla}^2 f = (K-\lambda)\hat{\mathrm{g}} - \frac{1}{2}\mathrm{d}f\otimes\mathrm{d}f - \frac{1}{2}\Phi^*\mathrm{g}_{\mathbb{H}^2}. \end{cases}$$

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$$\mathrm{g} \;=\; \pi^*\hat{\mathrm{g}}(x) + G_{ij}(x)\,\mathrm{d}\vartheta^i\mathrm{d}\vartheta^j, \qquad \vartheta^i \in \mathbb{R}/2\pi\mathbb{Z},$$

where $\hat{\mathrm{g}}$ is the quotient metric.

For $x \in \Sigma^\circ$, the area of the orbit $\mathsf{T}^2 = \pi^{-1}(x)$ is $4\pi^2\rho(x)$, where $\rho = \sqrt{\det G}$. Setting $\Phi = \rho^{-1}G$ so that $\det(\Phi) = 1$, we get a **weighted harmonic map** $\Phi : \Sigma^\circ \to \mathbb{H}^2$, i.e., a critical point of

$$\frac{1}{2}\int_{\Sigma^\circ} |\mathrm{d}\Phi|^2\,\rho\,\mathrm{d}v_{\hat{\mathrm{g}}}.$$

With $f = \log\rho$,

$$\mathrm{Ric}(\mathrm{g}) = \lambda\mathrm{g} \iff \begin{cases} \tau_f(\Phi) := \mathrm{tr}_{\hat{\mathrm{g}}}\,\nabla\mathrm{d}\Phi + \mathrm{d}\Phi(\hat{\nabla}f) = 0, \\ \Delta f + |\mathrm{d}f|^2 = -2\lambda, \\ \hat{\nabla}^2 f = (K-\lambda)\hat{\mathrm{g}} - \frac{1}{2}\mathrm{d}f\otimes\mathrm{d}f - \frac{1}{2}\Phi^*\mathrm{g}_{\mathbb{H}^2}. \end{cases}$$

These weighted harmonic maps exhibit interesting behavior...

The Weyl curvature

We can write $\mathrm{g} = \hat{\mathrm{g}}(x) + G_{ij}(x)\mathrm{d}\vartheta^i\mathrm{d}\vartheta^j$, with $\rho = \sqrt{\det G}$ and a weighted harmonic map $\Phi : \Sigma^\circ \to \mathbb{H}^2$. Let $\omega_{\mathcal{H}} := \mathrm{d}v_{\hat{\mathrm{g}}}$ and $\omega_{\mathcal{V}} := \star\omega_{\mathcal{H}}$. Then $\Phi^*\omega_{\mathbb{H}^2} = \mathrm{Jac}_\Phi \mathrm{d}v_{\hat{\mathrm{g}}}$.

Proposition. The 2-forms $\Omega_\pm := \frac{1}{\sqrt{2}}(\omega_{\mathcal{H}} \pm \omega_{\mathcal{V}})$ are eigenforms of the self-dual/anti-self-dual Weyl curvature $\mathcal{W}^\pm$ with eigenvalues

$$\mu_\pm = K_{\hat{\mathrm{g}}} \pm \frac{1}{2}\mathrm{Jac}_\Phi - \frac{\lambda}{3},$$

where $K_{\hat{\mathrm{g}}}$ is the Gauss curvature of $\hat{\mathrm{g}}$.

In particular, the eigenvalues of $\mathcal{W}^\pm$ can be studied by objects that are amenable to the Bochner technique, namely **weighted harmonic maps**. As a consequence, a delicate maximum principle argument yields the following.

Proposition. If a T^2-invariant Einstein metric is not half-conformally-flat (i.e., $\mathcal{W}^\pm \not\equiv 0$), then $\mu_\pm < 0$.

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Proposition. If a T^2-invariant Einstein metric is not half-conformally-flat (i.e., $\mathcal{W}^\pm \not\equiv 0$), then $\mu_\pm < 0$.

From $\mu_\pm \leq 0$, we are able to **significantly constrain** the T^2-actions that can admit invariant Einstein metrics.

If $E_1, ..., E_N$ denote the edges of the orbit space Σ, the preimages $\mathcal{D}_k := \pi^{-1}(\overline{E}_k)$ are **totally geodesic spheres** in M with self-intersection number

$$\mathcal{D}_k^2 \quad = \quad \frac{1}{2\pi}\int_{\mathcal{D}_k} \Theta_k^\perp dA, \qquad \Theta_k^\perp := \mathrm{g}(\mathrm{R}(e_1, e_2)\nu_2, \nu_1),$$

where dA is the area form on $\mathcal{D}_k \hookrightarrow (M, \mathrm{g})$, (e_1, e_2) is a local ON-frame for $T\mathcal{D}_k$ and (ν_1, ν_2) is a local ON-frame for its normal bundle, compatible with the orientation of M.

Theorem. If (M, g) is a closed simply connected 4-manifold with a T^2-invariant Einstein metric, then $\mathcal{D}_k^2 \in \{-1, 0, 1\}$.

- $3|\tau(M)| \leq \chi(M)$, and hence, there are no T^2-invariant Einstein metrics on $\sharp_2\mathbb{CP}^2$ or $\sharp_3\mathbb{CP}^2$.
- On $\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2}$ and $\sharp_m(S^2 \times S^2)$ (for each fixed $m \geq 1$), there is only one T^2-action that can admit an invariant Einstein metric.

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Uniqueness results

Theorem. Every Einstein metric admitting an isometric T^2-action on

$$S^4, \qquad \mathbb{CP}^2, \qquad S^2 \times S^2, \qquad \mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$$

is, up to scaling and isometry, respectively the round metric, the Fubini–Study metric, the product metric, or the Page metric.

The uniqueness results on S^4 and $\mathbb{CP}^2$ involve delicate maximum principle arguments and understanding the critical points of the weighted harmonic map. These type of arguments go back to **Schoen–Yau (1978)**.

For $S^2 \times S^2$ and $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$, we use the maximum principle to force $\mathcal{W}^{\pm}$ to have a repeated eigenvalue, using ideas of **Derdzinski (1983)** and **Peng Wu (2021)**.

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Comparison with the cohomogeneity-one classification

For the $\mathsf{SU}(2)$- and $\mathsf{SO}(3)$-actions, write

$$\mathrm{g} = \mathrm{d}t^2 + f_1^2\sigma_1^2 + f_2^2\sigma_2^2 + f_3^2\sigma_3^2,$$

where $\mathrm{d}\sigma_i = -2\sigma_j \wedge \sigma_k$ and (i,j,k) is cyclic.

Set $\gamma_i := f_i/(f_j f_k)$. Then $\mathrm{Ric}(\mathrm{g}) = \lambda \mathrm{g}$ becomes

$$\begin{cases} -\sum_{\ell=1}^{3} \dfrac{\ddot{f}_\ell}{f_\ell} = \lambda, \\ \dfrac{\ddot{f}_i}{f_i} + \dfrac{\dot{f}_i}{f_i}\left(\dfrac{\dot{f}_j}{f_j} + \dfrac{\dot{f}_k}{f_k}\right) = 2\gamma_i^2 - 2(\gamma_j - \gamma_k)^2 - \lambda. \end{cases}$$

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The induced equations on Λ^+ and Λ^-

Set $e^0 := \mathrm{d}t$ and $e^i := f_i \sigma_i$. The forms

$$\Omega_i^{\pm} := e^0 \wedge e^i \pm e^j \wedge e^k$$

are eigenforms of the curvature operator $\mathfrak{R} : \Lambda^{\pm} \to \Lambda^{\pm}$.

Write $\alpha_i^{\pm} = -A_i^{\pm} e^i$ for the induced connection forms and $\mathfrak{R}(\Omega_i^{\pm}) = a_i^{\pm} \Omega_i^{\pm}$, where

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Existence

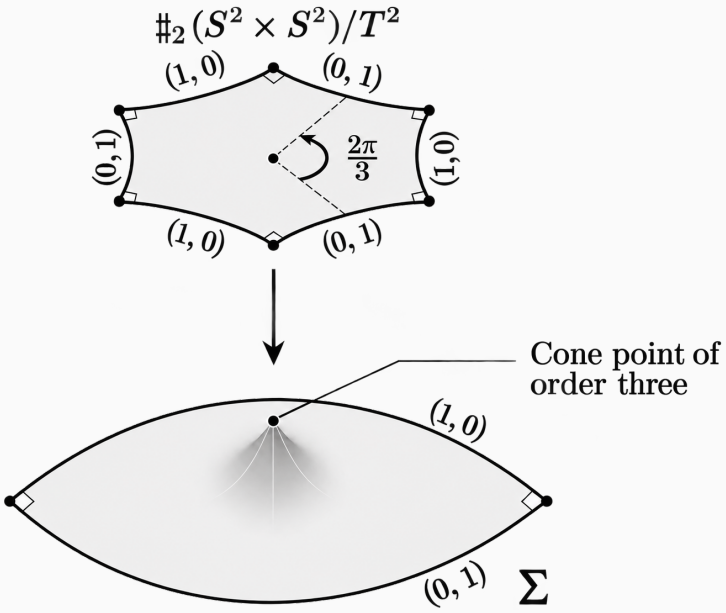
Theorem. (B.–Lafuente, Oct 2026). For every $m \geq 1$, there exists a T^2-invariant Einstein metric on $\sharp_m(S^2 \times S^2)$.

The behavior of the weighted harmonic map points to a branched covering.

Fix a principal T^2-orbit $\mathcal{T}$ of the T^2-action on S^4. For each $m \geq 1$, there is an equivariant degree $m+1$ covering

$$\sharp_m(S^2 \times S^2) \longrightarrow S^4$$

branched along $\mathcal{T}$.



Fine–Premoselli constructed Einstein metrics with negative sectional curvature on branched covers using a gluing construction.

Our approach is to construct a T^2-invariant Einstein metric on $S^4 \setminus \mathcal{T}$ with cone angle $2\pi/(m+1)$ along $\mathcal{T}$.

The use of conical metrics to produce smooth Einstein metrics is well-established (**Chen–Donaldson–Sun**, **Tian**). Previous conical arguments involved deforming the cone angle $2\pi\beta \to 2\pi$.

We deform the cone angle **downward**, from π to $2\pi/(m+1)$.

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A further contrast with existing conical methods is that the main work of our proof is to establish the *existence of the conical Einstein metric*.

The most natural approaches to construct the conical metric is via the *continuity method* (**Aubin, Chen–Donaldson–Sun, Tian, Yau**) or via *degree theory* (**Anderson**).

We established the existence of the conical metric using degree theory, analogous to Anderson's work on the existence of conformally compact Einstein metrics with prescribed conformal infinity.

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$$\mathrm{ind}(A) = \dim\ker A - \dim\mathrm{coker}(A).$$

We use the following consequence of **Smale's degree theory (1965)**.

Proposition. For $\beta \in [\beta_0, \frac{1}{2}]$, let $\mathcal{E}_\beta(g) = 0$ denote the Einstein equation in our chosen gauge and symmetry class, with cone angle $2\pi\beta$ along $\mathcal{T} \subset S^4$. Suppose:

1. At every solution, the linearized Einstein operator is Fredholm of index zero.
2. The total solution set $\{(\beta, g)\}$ is compact.
3. At $\beta = \frac{1}{2}$, there is exactly one Einstein metric, and its linearized operator is invertible.

Then an Einstein metric exists for every cone angle $2\pi\beta \in [2\pi\beta_0, \pi]$.

3 is handled by the **uniqueness** and **infinitesimal rigidity** of the product metric on $S^2 \times S^2$. Indeed, for $\beta = \frac{1}{2}$, the conical Einstein metric lifts to a smooth Einstein metric on $S^2 \times S^2$.

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Proposition. For $\beta \in [\beta_0, \frac{1}{2}]$, let $\mathcal{E}_\beta(g) = 0$ denote the Einstein equation in our chosen gauge and symmetry class, with cone angle $2\pi\beta$ along $\mathcal{T} \subset S^4$. Suppose:

1. At every solution, the linearized Einstein operator (in the gauge) has index zero.
2. The total solution set $\{(\beta, g)\}$ is compact.
3. At $\beta = \frac{1}{2}$, there is exactly one Einstein metric, and the linearized operator is invertible.

Then an Einstein metric exists for every cone angle $2\pi\beta \in [2\pi\beta_0, \pi]$.

3 is handled by the **uniqueness** and **infinitesimal rigidity** of the product metric on $S^2 \times S^2$. Indeed, for $\beta = \frac{1}{2}$, the conical Einstein metric lifts to a smooth Einstein metric on $S^2 \times S^2$.

A bounded linear operator $A : \mathcal{X} \to \mathcal{Y}$ between Banach spaces is **Fredholm** if $\mathrm{Im}(A) \subseteq \mathcal{Y}$ is closed with $\dim\ker(A) < \infty$ and $\dim\mathrm{coker}(A) < \infty$. Its **index** is

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We know that $\mathrm{g} = \hat{\mathrm{g}}(x) + G_{ij}(x)\mathrm{d}\vartheta^i\mathrm{d}\vartheta^j$. The model cone metric with cone angle $2\pi\beta$ along $\mathcal{T}$ is given by

$$\mathrm{g}_{\mathrm{edge}} \quad = \quad x^{2\beta}\left(\left(\frac{\mathrm{d}x}{x}\right)^2 + \mathrm{d}\vartheta^2\right) + \mathrm{g}_{\mathcal{T}},$$

where $\mathrm{g}_{\mathcal{T}}$ is a translation-invariant flat metric on the singular orbit.

Because the singular locus of the conical metrics is a single T^2-orbit, invariance removes the need to consider derivatives along $\mathcal{T}$.

After choosing isothermal coordinates, the metric is determined by three functions f, u, v on $\Sigma^\circ \setminus \{\pi(\mathcal{T})\}$:

$$\mathrm{g} \quad = \quad e^{2v}x^{2\beta}\mathrm{g}_b + e^{f+u}\,d\vartheta_1^2 + e^{f-u}\,d\vartheta_2^2.$$

The Fredholm properties required for the degree argument then involve specifying suitable function spaces for f, u, v.

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The Fredholm properties required for the degree argument then involve specifying suitable function spaces for f, u, v.

The compactness argument is the most technical. We need to show that a sequence of conical Einstein metrics g_i with cone angle $2\pi\beta_i$ admits a subsequence converging to a conical Einstein metric of the same type. This requires several things:

Uniformly bounded curvature. Suppose the curvature is unbounded $\sup |\mathrm{Rm}(g_i)|(p_i) \to \infty$. Then rescale the metrics so that $|\mathrm{Rm}(\tilde{g}_i)|(p_i) = 1$. We then show that the metrics $\tilde{g}_i$ converge to a complete non-flat Ricci-flat metric.

If the conical orbit remains at bounded distance, the limiting normalized orbit-area density is harmonic and attains its maximum at the cone point. The maximum principle then forces ρ to be constant, which then forces flatness.

If $\mathrm{dist}_{\tilde{g}_i}(p_i, \mathcal{T}) \to \infty$, the limit is smooth. In this case, we use the topology of the orbit space and the Einstein equations to bound the areas of the torus orbits. Completeness and nonpositive curvature of the orbit space then force these areas to be constant, and the remaining equations force the metric to be flat.

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The uniform curvature bound then establishes volume non-collapsing, since

$$64\pi^2 \quad = \quad \int_{S^4 \setminus \mathcal{T}} |\mathrm{Rm}(g)|_g^2 \mathrm{d}v_g \leq C^2 \mathrm{vol}(g).$$

Diameter bounds and comparison theorems then allow us to establish smooth subsequential convergence away from the conical orbit.

We must also ensure that the cone point remains uniformly separated from $\partial\Sigma$ and that the limiting metric is T^2-invariant. The separation follows from the curvature bound and Jacobi comparison along a geodesic from the cone point to the boundary. Uniform bounds for the two Killing fields generating the torus action allow this action to pass to the limit.

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Further problems

The diffeomorphism types of closed simply connected 4-manifolds admitting a T^2-invariant Einstein metric are fully classified.

1. For each fixed $2 \leq p \leq q \leq 2p$, there are still finitely many T^2-actions on $\sharp_p \mathbb{CP}^2 \sharp_q \overline{\mathbb{CP}^2}$ for which the existence of an invariant Einstein metric has not yet been established.
2. Uniqueness has only been established for S^4, $S^2 \times S^2$, and $\mathbb{CP}^2 \sharp k \overline{\mathbb{CP}^2}$ for $0 \leq k \leq 3$.