MEASURE THEORY LECTURE 4

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Some Motivation for Abstract Measure Theory. Riemann's theory of integration does not provide a well-behaved theory of integration with respect to limit operations. For instance, suppose $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ are a sequence of functions converging (in some sense) to some $f : \mathbb{R} \longrightarrow \mathbb{R}$, then one would like

$$\lim_{n \longrightarrow +\infty} \int f_n = \int \lim_{n \longrightarrow +\infty} f_n.$$

Unfortunately, this is not true unless the convergence of $f_n \longrightarrow f$ is in a very strong sense (e.g., uniform convergence).

This has significant ramifications for an analysis of function spaces (i.e., the study of Banach and Hilbert spaces – infinite-dimensional vector spaces endowed with a *complete* inner product or norm). In turn, this limits the utility of function spaces defined by some Riemann integrability criterion in the theory of partial differential equations (PDE).

More precisely, if we consider the space $L^2([0,1])$ of functions satisfying

$$\|f\|^2_{L^2([0,1])} := \int_0^1 |f(x)|^2 dx < +\infty,$$

then this space is not complete (in the sense that Cauchy sequences converge) with respect to the norm $\| \cdot \|_{L^2([0,1])}$ if the integral is understood as a Riemann integral.

It is complete, however, if the integral is understood to be the *Lebesgue integral*. One now encounters (at least) two problems:

(i) How does one construct such an integral?
(ii) Does this new integral honestly generalize the Riemann integral (which enjoys many desirable properties such as the fundamental theorem of calculus)?

There is motivation to consider measures/integrals beyond just the Lebesgue measure. For instance, the spectral theorem for unbounded self-adjoint operators requires the theory of projection-valued measures – this is essential for quantum mechanics. In fact, the most commonly occuring infinite-dimensional Hilbert space that occurs in quantum mechanics is $L^2(\mathbb{R}^3)$.

1. Some Reminders

Fix a universal set R. Denote by $\mathcal{R}$ a collection of sets (in R).

Definition 1.1. We declare $\mathcal{R}$ to be a *ring* if it is non-empty and

(i) is closed under unions, i.e., $A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R}$;
(ii) is closed under complements, i.e., $A, B \in \mathcal{R} \implies A \backslash B \in \mathcal{R}$.

If, in addition, $R \in \mathcal{R}$, then we say that $\mathcal{R}$ is an *algebra*.

Example 1.2. It is clear that $\mathcal{R} := \{\text{open intervals in } \mathbb{R}\}$ violates (ii), and hence does not define a ring. Indeed, let $A = \mathbb{R} \in \mathcal{R}$ and $B = (0,1)$, then $A \backslash B = (-\infty, 0] \cup [1, \infty)$, which is obviously not an open interval. The same can be said for the set of closed intervals in $\mathbb{R}$.

Remark 1.3. Do these definitions coincide with the definitions of ring and algebra from abstract algebra?

Definition 1.4. Let $\mathcal{R}$ be a ring. We say that $\mathcal{R}$ is a σ*−ring* if

$$(A_n)_{n \in \mathbb{N}} \implies \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{R}.$$

If, in addition, $\mathcal{R}$ contains the universal set R, then we declare $\mathcal{R}$ to be a σ*−algebra.*

Aside: Why the terminology "σ−algebra"? In (point-set) topology, an F_σ set is a countable union of closed sets. The F comes from the French word *fermé* (meaning closed) and σ being Greek for S which is the leading letter in the French word *somme* (meaning sum or union). Hence, the letter σ is used to indicate countable unions. A σ−algebra is a collection of subsets that is stable under countable unions. The analog for a countable intersection of open sets is called a G_δ set.

Example 1.5. A dumb example of a σ−algebra is $\{\phi, R\}$. A slightly less-trivial example is given by the power set of R.

Theorem 1.6. Let $\mathcal{U}$ be a collection of non-empty open subsets of R. There exists a unique ring $\mathcal{R}(\mathcal{U})$ that contains $\mathcal{U}$ (as a proper subset[1]) and is contained in every ring that contains $\mathcal{U}$.

Definition 1.7. The ring $\mathcal{R}(\mathcal{U})$ in Theorem 1.6 is called the *ring generated by* $\mathcal{U}$.

Definition 1.8. A map $\mu : \mathcal{R} \longrightarrow [0, +\infty]$ is a *measure* if

(i) $\mu(\emptyset) = 0$;
(ii) (σ−additivity). For $(A_n)_{n \in \mathbb{N}} \subset \mathcal{R}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, and $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{R}$, the equality

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

holds.

Remarks 1.9. If $\mu(A \cup B) = \mu(A) + \mu(B) = \mu(A) + \mu(B)$, then $\mu(\emptyset) = \mu(\emptyset \cup \emptyset) = 2\mu(\emptyset)$, and thus $\mu(\emptyset) = 0$.

[1] Otherwise the statement is trivial.

Proposition 1.10. A measure is *monotone* in the sense that $A \subset B$ implies $\mu(A) \leq \mu(B)$.

Proof. Write $B = A \cap B + B \backslash A$. Then

$$\mu(B) = \underbrace{\mu(A \cap B)}_{=\mu(A)} + \underbrace{\mu(B \backslash A)}_{\geq 0} \geq \mu(A).$$

□

2. Outer Measures

From now on, we assume that $\mu(R) < \infty$.

Definition 2.1. Let $\mathcal{R} \subset 2^R$ be an algebra. Endow $\mathcal{R}$ with a measure $\mu : \mathcal{R} \longrightarrow [0, +\infty]$. Define the *outer measure* (of μ) to be the function $\mu^* : 2^R \longrightarrow \mathbb{R}$ given by

$$\mu^*(A) := \inf \sum_{i=1}^{n(\infty)} \mu(E_i),$$

where the infimum is taken over all coverings (possibly infinite) of A by $(E_\alpha)_{\alpha \in A}$, for some indexing set A.

Proposition 2.2. Let $\mu^* : 2^R \longrightarrow \mathbb{R}$ be the outer measure of a measure $\mu : \mathcal{R} \longrightarrow [0, +\infty]$, where $\mathcal{R}$ is an algebra. If $A \in \mathcal{R}$, then the value of the outer measure on A coincides with that of the measure itself, i.e.,

$$A \in \mathcal{R} \implies \mu^*(A) = \mu(A).$$

Remark 2.3. By definition, the outer measure μ^* is non-negative. From Proposition, we see that $\mu^*(\emptyset) = 0$. In general, however, μ^* will not be $\sigma-$additive, and thus not define a measure, in general. It does, however, always satisfy $\sigma-$subadditivity:

$$\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu(A_n).$$

3. Measurable Sets

Definition 3.1. A set $A \subset R$ is said to be *measurable* if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

for each $E \subset R$.

Cautionary remark 3.2. Note that in the lecture notes, Artem uses the notation $\overline{A} := R \backslash A$ to indicate the set-theoretic complement (not the closure of a set).

Theorem 3.3. Let $\mu : \mathcal{R} \longrightarrow [0, +\infty]$ be a measure on an algebra $\mathcal{R}$. Let $\widetilde{\mathcal{R}} \subset R$ denote the set of measurable sets. The restriction of the outer measure μ^* to $\widetilde{\mathcal{R}}$ defines a measure, which we denote by

$$\widetilde{\mu} := \mu^*|_{\widetilde{\mathcal{R}}}.$$

For the statement of Theorem to be meaningful, we need to show that

(i) $\widetilde{\mathcal{R}}$ is a $\sigma-$algebra;
(ii) $\widetilde{\mu} : \widetilde{\mathcal{R}} \longrightarrow [0, +\infty]$ is $\sigma-$additive;
(iii) $\mathcal{R}$ is contained in $\widetilde{\mathcal{R}}$.

Lemma 3.4. $\widetilde{\mathcal{R}}$ is a $\sigma-$algebra.

Proof. We need to show that $\widetilde{\mathcal{R}}$ is closed under both finite and countable unions, complements, and R is contained in $\widetilde{\mathcal{R}}$.

(i) Let $A, B \in \widetilde{\mathcal{R}}$. We want to show that $A \cup B \in \widetilde{\mathcal{R}}$, i.e., for any $E \subset R$, we have

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

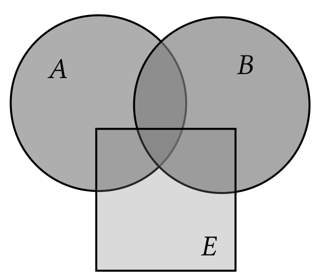
Since $A \in \widetilde{\mathcal{R}}$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

and since $B \in \widetilde{\mathcal{R}}$, we have

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c).$$

Now consider the picture



From which, we see that

$$\mu^*(E \cap (A \cup B)) = \mu^*(E \cap A) + \mu^*(E \cap B) - \mu^*(E \cap A \cap B),$$

and

$$\mu^*(E \cap (A \cup B)^c) = \mu(E) - \mu^*(E \cap A) - \mu^*(E \cap B) + \mu^*(E \cap A \cap B).$$

Hence,

$$\begin{aligned}\mu^*(E\cap(A\cup B))+\mu^*(E\cap(A\cup B)^c) &= \mu(E)-\mu^*(E\cap A)-\mu^*(E\cap B)+\mu^*(E\cap A\cap B)\\ &\quad +\mu^*(E\cap A)+\mu^*(E\cap B)-\mu^*(E\cap A\cap B) = \mu(E).\end{aligned}$$

(ii) Now we need to show that for $A, B \in \widetilde{\mathcal{R}}$, we have $A\backslash B \in \widetilde{\mathcal{R}}$. Since the definition of measurability is invariant under complements, this is immediate.

(iii) Hence, it remains to show that if $(A_k)_{k\in\mathbb{N}} \in \widetilde{\mathcal{R}}$, then $\bigcup_{k\in\mathbb{N}} A_k \in \widetilde{\mathcal{R}}$. Since μ^* is σ−subadditive, we know that for any $E \subset R$, we have

$$\mu^*(E) \le \mu^*\left(E\cap\bigcup_{k\in\mathbb{N}} A_k\right)+\mu^*\left(E\cap\left(\bigcup_{k\in\mathbb{N}} A_k\right)^c\right).$$

We need only show that

$$\mu^*(E) \ge \mu^*\left(E\cap\bigcup_{k\in\mathbb{N}} A_k\right)+\mu^*\left(E\cap\left(\bigcup_{k\in\mathbb{N}} A_k\right)^c\right)$$

in order to achieve equality.

From before, we know that for any fixed $n \in \mathbb{N}$, the union $\bigcup_{k=1}^n A_k \in \widetilde{\mathcal{R}}$. So

$$\begin{aligned}\mu^*(E) &= \mu^*\left(E\cap\bigcup_{k=1}^n A_k\right)+\mu^*\left(E\cap\left(\bigcup_{k=1}^n A_k\right)^c\right)\\ &\ge \sum_{k=1}^n \mu^*(E\cap A_k)+\mu^*\left(E\cap\left(\bigcup_{k=1}^n A_k\right)^c\right),\end{aligned}$$

from the subadditivity of the outer measure. From the monotonicity of the outer measure, we also see that

$$\begin{aligned}\mu^*\left(E\cap\left(\bigcup_{k=1}^n A_k\right)^c\right) = \mu^*\left(E\cap\bigcap_{k=1}^n A_k^c\right) &\ge \mu^*\left(E\cap\bigcap_{k\in\mathbb{N}} A_k^c\right)\\ &= \mu^*\left(E\cap\left(\bigcup_{k\in\mathbb{N}} A_k\right)^c\right).\end{aligned}$$

Allowing $n \longrightarrow \infty$, we have

$$\mu^*(E) \ge \sum_{k=1}^\infty \mu^*(E\cap A_k)+\mu^*\left(E\cap\left(\bigcup_{k=1}^n A_k\right)^c\right).$$

By the subadditivity of the outer measure again, however,

$$\sum_{k=1}^\infty \mu^*(E\cap A_k) \ge \mu^*\left(\bigcup_{k=1}^\infty E\cap A_k\right) = \mu^*\left(E\cap\bigcup_{k\in\mathbb{N}} A_k\right).$$

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