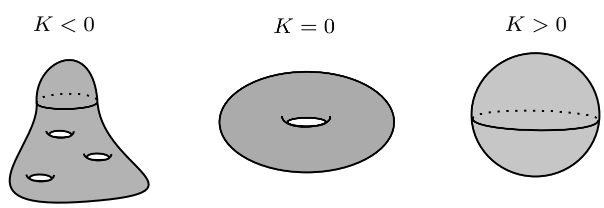
KYLE BRODER – RESEARCH STATEMENT

One of the most captivating aspects of mathematics lies in the serendipitous discovery of profound connections between seemingly unrelated fields. Often, two distinct areas that initially appear to have no relation unexpectedly reveal themselves to be intimately linked, different facets of a deeper underlying structure. As mathematicians, we hope to unearth such connections and analogies to transfer tools, techniques, and insights from one field to the other field.

My research is centered around leveraging *curvature* in differential geometry to understand fundamental concepts in algebraic, complex, and arithmetic geometry.

A perfect exemplification of this is the uniformization theorem for compact Riemann surfaces. A compact oriented surface Σ_g of genus g admits a conformal structure determined by the sign of its Gauss curvature K. Indeed, the curvature illuminates three model geometries: The *projective line* $\Sigma_0 \simeq \mathbf{CP}^1 \iff K > 0$; *elliptic curves* $\Sigma_1 \simeq \mathbf{C}/\Lambda \iff K = 0$; *hyperbolic surfaces* $\Sigma_{g \geq 2} \simeq \mathbf{D}/\Gamma \iff K < 0$.



These model geometries admit several higher-dimensionsional generalizations and have arisen in different fields. Many of the significant open problems and conjectures have concerned the relation between these various generalizations.

1. **Kobayashi Hyperbolic Manifolds**

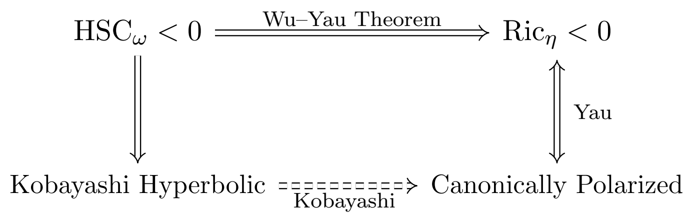
Hyperbolic Riemann surfaces $\Sigma = \Sigma_{g \geq 2}$ can be characterized by every holomorphic map from $\mathbf{C}$ into Σ being constant. On the other hand, it goes back to Poincaré that Σ can be realized as an algebraic curve in projective space $\mathbf{CP}^N$ via so-called *pluricanonical forms* $f(z)dz^{\otimes \ell}$; in particular, Σ is locally described by the roots of homogeneous polynomials. The former is the defining property of *Kobayashi hyperbolic* manifolds, while the latter is the defining property of *canonically polarized* manifolds.

A long-standing conjecture made by S. Kobayashi more than 50 years ago is the following relation between these notions:

The Kobayashi Conjecture. A compact Kobayashi hyperbolic manifold is projective and canonically polarized.

The Kobayashi conjecture has received considerable attention due to the recent breakthrough by D. Wu and S. T. Yau (2016) using curvature to bridge these notions. Indeed, an important class of Kobayashi hyperbolic manifolds are those with Hermitian metrics of negative *Holomorphic Sectional Curvature* $\text{HSC}_\omega < 0$. On the other hand, by Yau's resolution of the Calabi conjecture (1976), a compact complex manifold is canonically polarized if and only if it admits a Kähler metric (a very well-behaved class of Hermitian metrics) with negative *Ricci curvature* $\mathrm{Ric}_\eta < 0$.

The Wu–Yau Theorem. A compact Kähler manifold with a *Kähler* metric of $\mathrm{HSC}_\omega < 0$ is projective and canonically polarized.



Main Technique: The Schwarz Lemma. The main technique in the proof of the Wu–Yau theorem is the Schwarz lemma. This is the primary mechanism for estimating the geometry of one metric in terms of the geometry of another. Yau has consistently underscored the significance of the Schwarz lemma, emphasizing that many of his results have been a consequence of effectively leveraging and exploiting the Schwarz lemma in various contexts.

A heuristic understanding of the Schwarz lemma is given by envisioning a rigid 'harmonic' object residing in elastic equilibrium on a space X. When this object is transferred onto another manifold Y, three factors determine whether the object will maintain elastic equilibrium: (1) the curvature of X, (2) the curvature of Y, and (3) how the object is placed on Y. Improvements on the Schwarz lemma come in the form of requiring weaker assumptions on the curvature of X or the curvature of Y.

I produced novel Schwarz-type estimates, and gave a unified perspective on all existing Schwarz lemmas in the following papers:

The Schwarz Lemma in Kähler and Non-Kähler Geometry. Available at arXiv:2109.06331. To appear in the *Asian Journal of Mathematics*.

The Schwarz Lemma: An Odyssey. Available at arXiv:2110.04989. Published in the *Rocky Mountain Journal of Mathematics*.

A sophisticated use of the Schwarz lemma for *singular* and *degenerate metrics* was applied in the context of the *Kähler–Ricci flow* in:

Second-Order Estimates for Collapsed Limits of Ricci-flat Kähler Metrics. Available at arXiv:2106.13343. Published in the *Canadian Mathematical Bulletin*.

The symmetries that a metric possesses often allow one to relax the requirements on the curvature conditions for the Schwarz lemma to hold. Kähler metrics are examples of very well-behaved metrics with several symmetries.

H. Royden (1980) showed that for *Kähler metrics*, the Holomorphic Sectional Curvature is sufficient for the Schwarz lemma. This is the version of the Schwarz lemma used in the proof of the Wu–Yau theorem.

For a non-Kähler Hermitian metric, this is not the case. A stronger curvature constraint, negative *Real Bisectional Curvature* $\mathrm{RBC}_\omega < 0$, introduced by X. Yang and F. Zheng (2017) was identified to be sufficient for the Schwarz lemma.

Main Question. For a general Hermitian metric, can the Real Bisectional Curvature RBC_ω be replaced by the more natural Holomorphic Sectional Curvature HSC_ω in the Schwarz lemma?

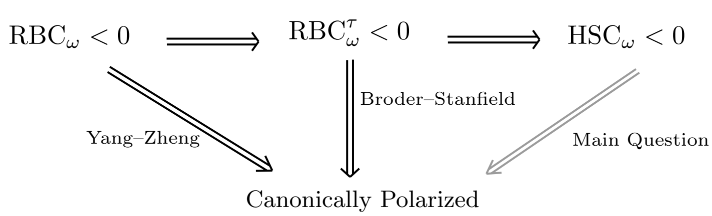
This is a very hard problem since the Real Bisectional Curvature is defined by the property of what allows the Schwarz lemma to work.

In recent joint work with James Stanfield, we discovered an *intermediate curvature condition* between the Holomorphic Sectional and Real Bisectional Curvatures that still permits the Schwarz lemma to work. We called this the *Tempered Real Bisectional Curvature* $\mathrm{RBC}_\omega^\tau < 0$. These results will appear in the following forthcoming paper:

A General Schwarz Lemma for Hermitian Manifolds. Joint with James Stanfield. In preparation.

An important consequence of our improved Schwarz lemma is the following general form of the Wu–Yau theorem in terms of the Tempered Real Bisectional Curvature:

Theorem. A compact Kähler manifold with a *Hermitian* metric with $\mathrm{RBC}_\omega^\tau < 0$ is projective and canonically polarized.



Generalizations of the Wu–Yau Theorem. An important aspect of the Wu–Yau theorem is understanding to what extent the condition $\mathrm{HSC}_\omega < 0$ can be relaxed. Diverio–Trapani (2016) showed that it suffices to assume that $\mathrm{HSC}_\omega \leq 0$ and $\mathrm{HSC}_\omega < 0$ at one point.

In recent joint work with Kai Tang, we showed that there is *no sign requirement* on the Holomorphic Sectional Curvature, our results permit $\mathrm{HSC}_\omega \leq \varepsilon$ globally, for $\varepsilon > 0$, and $\mathrm{HSC}_\omega \leq -\delta < 0$ on a local prescribed region. The results also extend to Hermitian metrics if the Holomorphic Sectional Curvature is replaced with the (Tempered) Real Bisectional Curvature. These results appear in:

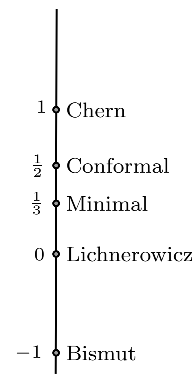
(ε, δ)–Quasi-Negativity and Positivity of the Canonical Bundle. Joint with Kai Tang. Available at arXiv:2305.01881.

These results build on earlier work that was carried out with Kai Tang:

On the Weighted Orthogonal Ricci Curvature. Joint with Kai Tang. Available at arXiv:2111.00346. Published in the *Journal of Geometry and Physics*.

2. The Gauduchon Connections

In differential geometry, derivatives of vector fields are made sense of using *connections*. In Riemannian and Kähler geometry, there is a canonical choice of connection – the *Levi-Civita connection*. For a general Hermitian metric, however, the Levi-Civita connection does not preserve the complex structure, and is most naturally replaced by the so-called *Chern connection*.



The space of Hermitian connections is infinite-dimensional, however, and it was discovered by Gauduchon (1997) that there is a natural one-dimensional line of connections. These *Gauduchon connections* ${}^t\nabla$, for $t \in \mathbf{R}$, pass through all known natural Hermitian connections that have appeared in both mathematics and physics. Most notably, the *Chern* (complex geometry) *Bismut* (heterotic string theory), *Lichnerowicz* (Riemannian geometry), *Minimal* (Hermitian geometry), and *Conformal* (conformal geometry) connections.

In joint work with James Stanfield, we gave a systematic treatment of the curvatures of these connections. We produced many new examples of *canonical Gauduchon–Einstein metrics* on *suspensions* of *Sasaki manifolds*. As a consequence of our results, we showed that many results that were previously believed to be a property of specific connections, held for all Gauduchon connections. We also discovered a very striking *monotonicity theorem* for the *Gauduchon Holomorphic Sectional Curvature*:

$$ {}^t\mathrm{HSC}_\omega \ \leq \ {}^c\mathrm{HSC}_\omega, $$

with equality if and only if $t = 1$ or the metric ω is Kähler. In particular, ${}^c\mathrm{HSC}_\omega < 0$ implies ${}^t\mathrm{HSC}_\omega < 0$ for all $t \in \mathbf{R}$. These results appear in:

On The Gauduchon Curvature of Hermitian Manifolds. Joint with James Stanfield. Available at arXiv:2211.05973. Published in *The International Journal of Mathematics*.

The results build off a general framework for the curvature of Hermitian manifolds that was carried out in the following earlier joint work with Kai Tang:

On the Altered Holomorphic Curvatures of Hermitian Manifolds. Joint with Kai Tang. Available at arXiv:2201.03666. Published in the *Pacific Journal of Mathematics*.

3. **Current Projects and Future Work**

1. (*Bridging the Kobayashi Conjecture and Wu–Yau theorems*). Demailly conjectured that compact Kobayashi hyperbolic manifolds are characterized by having *non-degenerate negative* k*–jet curvature*. With James Stanfield (*The University of Queensland*), we are working to express this curvature condition in the more familiar language of Riemannian geometry and attempting to prove a Wu–Yau-type theorem under this curvature condition. We have already made progress, proving the following Wu–Yau-type theorem under a slightly stronger curvature condition:

Theorem. A compact complex manifold with a metric of *non-degenerate negative total* k*–jet curvature* is projective and canonically polarized.

2. (*Prescribed Ricci Curvature Problem*). With my postdoctoral advisors Artem Pulemotov (*The University of Queensland*) and Wolfgang Ziller (*The University of Pennsylvania*), we have discovered a rigidity phenomenon exhibited by the second Chern Ricci curvature. These results both extend our understanding of the Wu–Yau theorem and address the question of existence of second Chern Einstein metrics on a large class of compact Hermitian manifolds.

3. (*Curvature characterizations of Oka manifolds*). I am working on proving a positive analog of the Wu–Yau theorem for more general classes of manifolds. This is related to my current work with Finnur Lárusson (*The University of Adelaide*), where we are attempting to characterize Oka manifolds (positive curvature analogs of Kobayashi hyperbolic manifolds) using curvature conditions. We have already discovered a curvature condition for so-called *convex projective manifolds*.

4. (*Degenerate Wu–Yau*). S.-T. Yau asked whether one could extend the Wu–Yau theorem to possibly degenerate metrics. Stefano Trapani and I are currently working on the so-called pseudo-Kähler Wu–Yau problem, using methods of pluripotential theory.

5. (*Curvature and Moduli*). In a current joint project with Peter Petersen (*The University of California Los Angeles*) and Timothy Buttsworth (*The University of Queensland*), we are working on understanding the Kobayashi hyperbolicity of non-compact complex manifolds. In particular, we are attempting to address the long-standing problem raised by N. Mok: *Does the bidisk* $\mathbf{D} \times \mathbf{D}$ *admit a complete Kähler metric with strictly negative curvature?*