

On the Theory of Stein Spaces.

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Introduction

In his memorable work [44], G. Mittag-Leffler proved that if one prescribes the principal parts of a meromorphic function on a domain in \mathbb{C} , then one can always find a meromorphic function having exactly those principal parts. This problem of existence of a meromorphic function with prescribed principal parts was later generalised to several complex variables by P. Cousin in [11] and has since been labeled the *first Cousin problem*.

Contrary to the one variable situation, the first Cousin problem is not always solvable for domain in $\mathbb{C}^{n>1}$. The simplest example of such a domain is the punctured bidisk in \mathbb{C}^2 , obtained from removing the origin $(0, 0)$ from the bidisk $\Delta^{(2)}(0, 1) := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\}$. It is a famous theorem of F. Hartogs [27] that any function which is holomorphic on the punctured bidisk can be extended to a function which is holomorphic on the entire bidisk. In particular, the punctured bidisk is an example of a domain in \mathbb{C}^2 which is *not a domain of holomorphy*.

In 1934, H. Cartan [6] observed that for any domain in \mathbb{C}^2 where the first Cousin problem was solvable, must necessarily have been a domain of holomorphy. Some three years later, K. Oka [50] had shown that on any domain of holomorphy in \mathbb{C}^n , the first Cousin problem was always solvable. For dimension $n > 2$ however, the converse is not true. Indeed, Cartan in [7] showed that the first Cousin problem was always solvable on the domain $\mathbb{C}^3 \setminus \{0\}$, and such a domain is clearly not a domain of holomorphy (see, e.g., [19, p. 130–133] for details).

A few years prior to the result of Mittag-Leffler, K. Weierstrass [65] had proven that if one prescribes the roots (with multiplicity) of an entire function on a domain in \mathbb{C} , then one could find an entire function having exactly those roots (with multiplicity). This problem was generalised to several complex variables again by Cousin and has since been referred to as the *second Cousin problem*. The second Cousin problem is substantially more difficult to solve, in comparison with the first Cousin problem, and is not always solvable on a domain of holomorphy.

It was his interest in the Cousin problems that lead Stein to introduce the class of manifolds in which now bear his name. In [62], Stein introduces, for domains \mathfrak{G} in complex manifolds \mathfrak{M}^{2n} , the following three axioms¹:

1. (*Holomorphic Convexity*). For every compact subset \mathfrak{G}_0 of \mathfrak{G} there is a compact subset \mathfrak{G}_1 which contains it so that for every point P in \mathfrak{G} which is not contained in \mathfrak{G}_1 there is a holomorphic function f_P on \mathfrak{G} with

$$|f_P(p)| > \text{Max}|f_P(K_0)|.$$

¹The following is a translation from the original German paper [62].

2. (*Point Separation*). For any two different points P_1 and P_2 in \mathfrak{G} there is a function f_{P_1, P_2} which is holomorphic on \mathfrak{G} and which takes on different values at P_1 and P_2 .
3. (*Coordinates*). For every Q in \mathfrak{G} there is a system of n holomorphic functions on \mathfrak{G} whose functional determinant at Q is non-zero.

In [62], Stein refers to the manifolds which satisfy these three axioms as *holomorphically complete*. In 1953 however, Cartan [10] in his reformulation of complex analysis based on sheaf cohomology baptised such spaces as *Variété de Stein*, and this language has remained.

Purpose and Structure of the Thesis. The purpose of this thesis is to detail this story and prove Cartan's theorem B on the sheaf-theoretic characterisation of Stein spaces. In more detail, the structure of the thesis is as follows.

1. Chapter 1 fixes the notation that will be used throughout the thesis and proves the extension theorem of Hartogs. We discuss the calculus of differential forms which is used both to prove the extension theorem of Hartogs and is used to define Dolbeault cohomology.
2. Chapter 2 introduces the definition of a Stein manifold, and some discussion of domains of holomorphy is given. We develop the necessary background on sheaves and introduce the general complex analytic spaces. The non-reduced complex analytic spaces will be the most general setting which we work in.
3. Chapter 3 develops sheaf cohomology, and in particular, sheaf cohomology via flabby and soft sheaves, Čech cohomology, and Dolbeault cohomology. Many of the proofs regarding sheaf cohomology are omitted in the interests of brevity. A statement of Cartan's theorem B is given in §3.3 and a proof is given for simply-connected polydomains in \mathbb{C}^n .
4. Chapter 4 is dedicated to the completion of the proof of Cartan's theorem B for general Stein spaces, and a discussion of the Cousin problems is given in §4.3.

No new results are presented in this thesis. The primary references used in this thesis are [18], [19], [24], [26], [37], and [59].

Further Remarks. Let us note the following choices that have been made in the synthesis of this thesis.

- If a proof has been omitted, a reference, with page number, is provided.
- If a result is attributed to a particular mathematician, or has some historical significance to the development of the theory, often the reference is given without page numbers being specified.
- An extensive list of notation has been provided to assist the reader.
- We have assumed that the reader is familiar with the areas of (one-variable) complex analysis and differential geometry. At times, we use some language from category theory, but no understanding beyond the very basic definitions is required here.

Notation

\mathbb{N}	The <i>natural numbers</i> . We do not assume that 0 is contained in \mathbb{N} .
\mathbb{N}_0	The set $\mathbb{N} \cup \{0\}$.
$\mathbb{R}_{>0}$	The <i>positive real numbers</i> , i.e., the set $(0, \infty)$.
$\mathbb{R}_{\geq 0}$	The <i>non-negative real numbers</i> , i.e., the set $[0, \infty)$.
\mathbb{R}_-	The set of <i>non-positive real numbers</i> , i.e., the set $(-\infty, 0)$.
i	$\sqrt{-1}$.
$\operatorname{Re}(z)$	The <i>real part</i> of a complex number z .
$\operatorname{Im}(z)$	The <i>imaginary part</i> of a complex number z .
$\arg(z)$	The <i>argument of a complex number</i> z .
$\log(z)$	The <i>logarithm of a complex number</i> z , $\log(z) := \log z + i\arg(z)$.
$\overset{\circ}{Q}$	The <i>topological interior</i> of a set Q .
\overline{U}	The <i>topological closure</i> of a set U .
Λ	An arbitrary indexing set.
\oplus	The <i>direct sum</i> .
\otimes	The <i>tensor product</i> .
\wedge	The <i>wedge product</i> .
\mapsto	“maps to”.
\twoheadrightarrow	Signifies that a map is <i>surjective</i> .
ι	An <i>inclusion map</i> .
$\Delta(w, r)$	The <i>disk of radius</i> $r \in \mathbb{R}_{>0}$ <i>centred at</i> $w \in \mathbb{C}$.
$\Delta^{(n)}(w, r)$	The <i>polydisk of polyradius</i> $r \in (\mathbb{R}_{>0})^n$ <i>centred at</i> $w \in \mathbb{C}^n$.
$\mathcal{C}(G)$	The set of <i>continous functions on</i> G .
$\mathcal{C}^k(G)$	The set of <i>k-times continuously differentiable functions on</i> G .
$\mathcal{C}^\infty(G)$	The set of <i>smooth functions on</i> G .
$\mathcal{C}_0^\infty(G)$	The set of <i>smooth functions with compact support in</i> G .
$\mathcal{O}(G)$	The set of <i>holomorphic functions (or maps) on</i> G .
$\Omega^k(G)$	The set of <i>smooth k-forms on</i> G .
$\Omega^{p,q}(G)$	The set of <i>smooth (p, q)-forms on</i> G .
$\operatorname{End}(V)$	The set of <i>endomorphisms of</i> V .
d	The <i>exterior derivative</i> .
$\bar{\partial}$	The <i>Dolbeault operator</i> .
$\operatorname{supp}(\rho)$	The <i>support of function</i> ρ , i.e., the closure of the set-theoretic support.
$\operatorname{sgn}(\sigma)$	The <i>sign of a permutation</i> σ .
$\operatorname{conv}(K)$	The <i>convex hull of</i> K .

$\text{Aff}(\mathbb{R}^n)$	The set of <i>affine functions</i> $T : \mathbb{R}^n \rightarrow \mathbb{R}$.
\widehat{K}	The <i>holomorphically convex hull</i> of K .
proj	A <i>projection map</i> .
res_U	<i>Restriction</i> to U , also denoted by $ _U$.
Id_X	The <i>identity map</i> on X .
$\text{im}(f)$	The <i>image</i> of f .
$\text{ker}(f)$	The <i>kernel</i> of f .
$\text{coker}(f)$	The <i>cokernel</i> of f .
$\mathcal{K}\text{er}(f)$	The <i>presheaf kernel</i> of f .
$\mathcal{I}\text{m}(f)$	The <i>presheaf image</i> of f .
$\mathcal{C}\text{oker}(f)$	The <i>presheaf cokernel</i> of f .
f_*	The <i>direct image</i> .
$V(f)$	The <i>zero set</i> of f .
$\mathbb{C}\{z\}$	The <i>ring of convergent power series centred at the origin</i> .
$\mathfrak{m}_{(z)}$	The <i>maximal ideal</i> in $\mathbb{C}\{z\}$.
\mathfrak{m}_p	The <i>maximal ideal</i> in \mathcal{O}_p .
\mathfrak{m}_p^k	The product $\underbrace{\mathfrak{m}_p \cdots \mathfrak{m}_p}_{k \text{ times}}$.
TM	The <i>tangent bundle</i> of a smooth manifold M .
$TM^{\mathbb{C}}$	The <i>complexified tangent bundle</i> of an almost complex manifold M .
T^*M	The <i>cotangent bundle</i> of a smooth manifold M .
$\Lambda(V)$	The <i>exterior algebra</i> of a vector space V .
$\mathcal{A}^{p,q}$	The <i>sheaf of smooth</i> (p, q) - <i>forms</i> .
$\mathcal{L}_G^{p,q}$	The <i>sheaf of $\bar{\partial}$-closed forms</i> on G . The subscript G is sometimes omitted.
$\mathcal{B}_G^{p,q}$	The <i>sheaf of $\bar{\partial}$-exact forms</i> on a domain G . The subscript G is sometimes omitted.
$\mathcal{D}^{p,q}(G)$	The <i>Dolbeault cohomology groups</i> , i.e., the quotient group $\mathcal{L}^{p,q}(G)/\mathcal{B}^{p,q}(G)$.
(S, π)	An <i>analytic stone</i> or an <i>analytic block</i> .
$\mathring{A}(S)$	The <i>analytic interior</i> of an analytic stone or analytic block S .
\preceq	An inclusion of analytic stones or analytic blocks.
\mathcal{N}_X	The <i>nilradical sheaf</i> .
X_{red}	The <i>reduction</i> of a complex analytic space X .
$\text{red } \mathcal{O}_X$	The <i>reduction</i> of the structure sheaf \mathcal{O}_X , i.e., $\text{red } \mathcal{O}_X := \mathcal{O}_X/\mathcal{N}_X$.
$\text{red}(f)$	The <i>reduction</i> of f .
\mathcal{O}^p	The direct sum $\mathcal{O}^p := \underbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}_{p \text{ times}}$.

CHAPTER 1

Analytic Functions of Several Complex Variables

The main purpose of this chapter is twofold. The first is to fix some notation, which is effectively all of §1.1. The second is to motivate the reader and to convince them that Stein spaces, which are introduced in Chapter 2, are of interest, and are worthwhile objects to study. In particular, §1.2 is devoted exclusively to the investigation of domains of convergence of power series in several complex variables, and in the final section, §1.4, we offer a proof of *Hartogs' extension theorem* (see, e.g., [27], [59, p. 172], [18, p. 307]), on compulsory analytic continuation. In §1.3 we include the calculus of differential forms required for the proof of Hartogs' extension theorem and the Dolbeault cohomology theory in §3.1.

§1.1. FORMALITIES, NOTATION, AND CONVENTIONS

We set up some notation that will be used throughout this exposition. Note also the table of notation at the beginning of the thesis.

Definition 1.1.1. The n -dimensional complex plane \mathbb{C}^n is defined to be the n -fold cartesian product of \mathbb{C} , i.e., $\mathbb{C}^n := \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n\text{-times}}$. The coordinates of \mathbb{C}^n are denoted by $z := (z_1, \dots, z_n)$, where $z_k := x_k + iy_k$, $x_k := \operatorname{Re}(z_k)$, $y_k := \operatorname{Im}(z_k) \in \mathbb{R}$ for each $1 \leq k \leq n$, and $i := \sqrt{-1}$.

Definition 1.1.2. The *complex conjugate* of $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ is denoted by $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n) \in \mathbb{C}^n$, where $\bar{z}_k := x_k - iy_k$ for each $1 \leq k \leq n$.

We endow \mathbb{C}^n with the *Hermitian scalar product* $(\cdot, \cdot) : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}$,

$$(z, w) := \sum_{j=1}^n z_j \bar{w}_j,$$

where $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$. This induces the standard *Euclidean norm* $|\cdot|$ on \mathbb{C}^n by setting $|z| := \sqrt{(z, z)}$.

Definition 1.1.3. The *open polydisk* in \mathbb{C}^n of *polyradius* $r = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$ centred at $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ is the product of open disks in \mathbb{C} , i.e.,

$$\Delta^{(n)}(w, r) := \{z \in \mathbb{C}^n : |z_j - w_j| < r_j, 1 \leq j \leq n\},$$

where $z = (z_1, \dots, z_n)$.

Notational Remark 1.1.4. In the one-dimensional case we omit the superscript (n) and write $\Delta(w, r)$ for the disk of radius $r \in \mathbb{R}_{>0}$ centred at $w \in \mathbb{C}$. Moreover, we permit each component radius r_k to be infinite and adopt the convention $\Delta(w, 0) := \emptyset$, and $\Delta(w, \infty) := \mathbb{C}$.

The *closed polydisk* in \mathbb{C}^n of polyradius $r = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$ centred at $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ is the product of the closed disks in \mathbb{C} , i.e.,

$$\overline{\Delta}^{(n)}(w, r) := \{z \in \mathbb{C}^n : |z_j - w_j| \leq r_j, 1 \leq j \leq n\}.$$

The convention in one dimension is maintained, and we write $\overline{\Delta}(w, r)$ for the closed disk of radius $r \in \mathbb{R}_{>0}$ centred at $w \in \mathbb{C}$.

Definition 1.1.5. A *domain* is a connected open subset of \mathbb{C}^n .

Notational Remark 1.1.6. A function $f : G \rightarrow \mathbb{C}$ defined on some open set $G \subseteq \mathbb{R}^n$ will be referred to as *smooth* if it is k -times continuously differentiable for all $k \in \mathbb{N}_0$.

The following function spaces will be frequently referred to throughout this exposition:

- $\mathcal{C}(G)$ denotes the set of all continuous functions on G .
- $\mathcal{C}^k(G)$ denotes the set of all k -times continuously differentiable functions on G .
- $\mathcal{C}^\infty(G)$ denotes the set of all smooth functions on G .

If the codomain of the map is to be specified, we will write $\mathcal{C}(G, Y)$, for example, which denotes the set of all continuous maps $f : G \rightarrow Y$. We reserve the term *function* for maps whose codomain is either \mathbb{R} or \mathbb{C} .

Definition 1.1.7. A *ring* is understood to mean a commutative ring with identity. If R is a ring, the identity element is denoted by 1_R .

Convention 1.1.8. We assume that all manifolds discussed in this thesis are second countable.

§1.2. POWER SERIES REPRESENTATIONS

In this section, we discuss the power series representations of holomorphic functions of several complex variables. Throughout this section, we let \mathbb{K} be a field, either \mathbb{R} or \mathbb{C} .

Definition 1.2.1. Let $G \subseteq \mathbb{C}^n$ be a domain. A function $f : G \rightarrow \mathbb{C}$ is said to be \mathbb{K} -*differentiable at a point* $w \in G$ if there exists a \mathbb{K} -linear function $df_w : \mathbb{C}^n \rightarrow \mathbb{C}$ and a function $\delta : G \rightarrow \mathbb{C}$ such that

$$f(z) = f(w) + df_w(z - w) + \delta(z)$$

for all $z \in G$, and $\delta(z)/|z - w| \rightarrow 0$ as $z \rightarrow w$. The function df_w is referred to as the *differential of f at w* . If f is \mathbb{C} -differentiable at every point in a neighbourhood of G , we say that f is *holomorphic* on G .

Definition 1.2.2. Let G be a domain in \mathbb{C}^n . A map $f = (f_1, \dots, f_m) : G \rightarrow \mathbb{C}^m$ is said to be \mathbb{K} -*differentiable at $w \in G$* if each component functions f_1, \dots, f_m is \mathbb{K} -differentiable at w . If f is \mathbb{C} -differentiable at every point in a neighbourhood of G , we say that f is *holomorphic* on G .

Notational Remark 1.2.3. The set of all holomorphic functions, or holomorphic maps, on a domain $G \subseteq \mathbb{C}^n$ is denoted by $\mathcal{O}(G)$. If the codomain is not understood from the context and needs to be specified, we will write $\mathcal{O}(G, Y)$ if, for example, the codomain is Y . We will also adopt the convention that for a closed set $K \subseteq \mathbb{C}^n$, by $f \in \mathcal{O}(K)$ we mean that f is the restriction to K of a function \tilde{f} which is holomorphic on an open neighbourhood of K .

Definition 1.2.4. Let D and G be two domains in \mathbb{C}^n . A holomorphic map $f : D \rightarrow G$ is said to be *biholomorphic* if f is bijective, holomorphic, and whose inverse $f^{-1} : G \rightarrow D$ is holomorphic.

It is a well-known result in complex analysis that if f is holomorphic and bijective, the inverse is automatically holomorphic.

Definition 1.2.5. Let $G \subseteq \mathbb{C}^n$ be a domain. Suppose that $f : G \rightarrow \mathbb{C}$ is \mathbb{R} -differentiable at $w \in G$. For each $1 \leq j \leq n$, $z \in \mathbb{C}^n$, we define

$$\left. \frac{\partial f}{\partial z_j} \right|_w := \frac{1}{2} \left(\left. \frac{\partial f}{\partial x_j} \right|_w - i \left. \frac{\partial f}{\partial y_j} \right|_w \right), \quad \left. \frac{\partial f}{\partial \bar{z}_j} \right|_w := \frac{1}{2} \left(\left. \frac{\partial f}{\partial x_j} \right|_w + i \left. \frac{\partial f}{\partial y_j} \right|_w \right)$$

Remark 1.2.6. We observe that for any \mathbb{R} -differentiable function f , and each $1 \leq j \leq n$, $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ satisfy

$$\overline{\left. \frac{\partial f}{\partial \bar{z}_j} \right|_w} = \left. \frac{\partial f}{\partial z_j} \right|_w.$$

Using the operators $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ we have the following *Cauchy–Riemann criterion* for \mathbb{C} -differentiability in each variable:

Definition 1.2.7. Let G be a domain in \mathbb{C}^n . We say that an \mathbb{R} -differentiable function $f : G \rightarrow \mathbb{C}$ is \mathbb{C} -differentiable with respect to the variable z_k at $w \in G$ if $\left. \frac{\partial f}{\partial \bar{z}_k} \right|_w = 0$. If this holds for each $1 \leq k \leq n$ then f is said to be *separately \mathbb{C} -differentiable* at $w \in G$. If f is separately \mathbb{C} -differentiable at all points in a neighbourhood of G we say that f is *separately holomorphic on G* .

Theorem 1.2.8. ([34, p. 9–10]). Let G be a domain in \mathbb{C}^n . If f is continuous on \overline{G} then f is holomorphic on G (in the sense of *Definition 1.2.1*) if and only if f is separately holomorphic on G .

Remark 1.2.9. *Theorem 1.2.8* is often referred to as the *Fundamental Theorem of Hartogs*. Further details on theorems of this type may be found in [34, Chapter 1]. We now consider the following higher-dimensional analogue of the Cauchy integral formula from one complex variable.

Theorem 1.2.10. ([59, p. 18]). Suppose that $f : \overline{\Delta}^{(n)}(w, r) \rightarrow \mathbb{C}$ is separately holomorphic on $\overline{\Delta}^{(n)}(w, r)$ and continuous on $\overline{\Delta}^{(n)}(w, r)$, then for all $z \in \Delta^{(n)}(w, r)$,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_n \cdots d\zeta_1,$$

where Γ_k denotes the boundary of $\Delta(w_k, r_k)$ for each k .

The following corollary will allow us to obtain a series representation which is the higher-dimensional analogue of the power series representation for holomorphic functions of a single complex variable.

Corollary 1.2.11. ([59, p. 18]). Suppose that $f : \overline{\Delta}^{(n)}(w, r) \rightarrow \mathbb{C}$ is holomorphic on $\Delta^{(n)}(w, r)$ and continuous on $\overline{\Delta}^{(n)}(w, r)$, then for all $z \in \Delta^{(n)}(w, r)$,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_n \cdots d\zeta_1.$$

The remainder of this section is devoted to the study of higher-dimensional power series, and their algebraic properties.

Definition 1.2.12. For $n \in \mathbb{N}$, a *multi-index* is an element of \mathbb{N}_0^n . For a multi-index $J = (j_1, \dots, j_n)$, we write $|J| := j_1 + \cdots + j_n$, and $J! := j_1! \cdots j_n!$. If $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, then $z^J := z_1^{j_1} \cdots z_n^{j_n}$. For two multi-indices $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$, we set $I + J := (i_1 + j_1, \dots, i_n + j_n)$, and $I \cdot J := (i_1 \cdot j_1, \dots, i_n \cdot j_n)$.

Definition 1.2.13. An expression of the form

$$\sum_{|J|=0}^{\infty} c_J (z - w)^J = \sum_{j_1 + \cdots + j_n = 0}^{\infty} c_{j_1 \cdots j_n} (z_1 - w_1)^{j_1} \cdots (z_n - w_n)^{j_n}$$

is called a *formal (multiple) power series centred at $w \in \mathbb{C}^n$* , where $c_J \in \mathbb{C}^n$, and $J = (j_1, \dots, j_n) \in \mathbb{N}_0^n$ is a multi-index.

The set of formal power series centred at w can be shown to be a ring which we denote by $\mathbb{C}[[z_1, \dots, z_n]](w)$. Indeed, if $f(z) = \sum_{|J|=0}^{\infty} a_J (z - w)^J$ and $g(z) = \sum_{|J|=0}^{\infty} b_J (z - w)^J$, then addition and multiplication in $\mathbb{C}[[z_1, \dots, z_n]](w)$ are defined by the respective formulae:

$$f(z) + g(z) := \sum_{|J|=0}^{\infty} (a_J + b_J) (z - w)^J, \quad \text{and} \quad f(z) \cdot g(z) := \sum_{k=0}^{\infty} \left(\sum_{|J+L|=k} a_J b_L \right) (z - w)^{J+L}.$$

Notational Remark 1.2.14. Observe that the change of coordinates $\zeta_j = z_j - w_j$ induces an isomorphism between $\mathbb{C}[[z_1, \dots, z_n]](w)$ and $\mathbb{C}[[\zeta_1, \dots, \zeta_n]](0)$. Hence, we often write $\mathbb{C}[[z_1, \dots, z_n]]$ for the ring of formal power series, implicitly assuming that all formal power series are centred at $0 \in \mathbb{C}^n$.

Definition 1.2.15. We define the *order of a formal power series* $f(z) := \sum_{|J|=0}^{\infty} c_J (z - w)^J$ to be the positive integer

$$\text{ord}(f) := \min \{|J| : c_J \neq 0, J \in \mathbb{N}_0^n\}.$$

If $c_J = 0$ for all $J \in \mathbb{N}_0^n$, then we define $\text{ord}(f) = \infty$.

Definition 1.2.16. A sequence of formal power series $(f_k)_{k \in \mathbb{N}}$ is said to be *summable* if for each $\ell \in \mathbb{N}$, there are only finitely many $k \in \mathbb{N}$ such that $\text{ord}(f_k) \leq \ell$.

We now want to discuss the convergence of (multiple) power series. Let us note that for $n > 1$, there is no canonical way of ordering the elements of \mathbb{N}_0^n . We remind ourselves that an *ordering* of \mathbb{N}_0^n is a bijection $\sigma : \mathbb{N}_0 \rightarrow \mathbb{N}_0^n$. For $\delta = (\delta_1, \dots, \delta_n) \in (\mathbb{R}_{>0})^n$ and σ an ordering of \mathbb{N}_0^n , we define a map $\|\cdot\|_{\delta, \sigma} : \mathbb{C}[[z_1, \dots, z_n]] \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$,

$$f(z) = \sum_{|J|=0}^{\infty} c_J(z-w)^J \mapsto \|f\|_{\delta, \sigma} := \sum_{j=0}^{\infty} |c_{\sigma(j)}| \delta^{\sigma(j)}.$$

Definition 1.2.17. A formal power series $f(z) = \sum_{|J|=0}^{\infty} c_J(z-w)^J$ is said to *converge (absolutely) at $z \in \mathbb{C}^n$ relative to the ordering σ* if $\delta \in (\mathbb{R}_{>0})^n$ can be chosen such that $\|f\|_{\delta, \sigma} < \infty$. The interior of the set of all $z \in \mathbb{C}^n$ such that $\|f\|_{\delta, \sigma} < \infty$ is called the *domain of convergence of f relative to the ordering σ* .

We have the following important result from first-semester calculus:

Lemma 1.2.18. Suppose that $f = \sum_{|J|=0}^{\infty} c_J(z-w)^J$ converges (absolutely) relative to the ordering σ , then f converges absolutely relative to any ordering in $\Delta^{(n)}(w, \delta)$.

Notational Remark 1.2.19. In the circumstances that $\|f\|_{\delta, \sigma}$ does not depend on the choice of ordering σ , we simply write $\|f\|_{\delta}$. In such circumstances, we assume the ordering is lexicographic.

Let us mention some useful properties of $\|\cdot\|_{\delta}$.

Lemma 1.2.20. ([15, p. 91]).

(i) For a summable sequence $(f_k)_{k \in \mathbb{N}}$ of formal power series, we have

$$\left\| \sum_{j=0}^{\infty} f_j \right\|_{\delta} \leq \sum_{j=0}^{\infty} \|f_j\|_{\delta}.$$

(ii) For formal power series f and g we have

$$\|f \cdot g\|_{\delta} \leq \|f\|_{\delta} \|g\|_{\delta}.$$

(iii) If $f(z) := \sum_{|J|=0}^{\infty} c_J z^J$ is convergent, then $\lim_{\delta \rightarrow 0} \|f\|_{\delta} = |f(0)|$.

Definition 1.2.21. Let G be the domain of convergence of the multiple power series $f(z) = \sum_{|J|=0}^{\infty} c_J(z-w)^J$. Suppose that $U \subseteq G$ is an open set and $\sigma : \mathbb{N}_0 \rightarrow \mathbb{N}_0^n$ is an ordering. We say that f *converges uniformly on U relative to σ* if the sequence of partial sums $\sum_{j=0}^K c_{\sigma(j)}(z-w)^{\sigma(j)}$ converges uniformly on U . If f converges uniformly relative to any choice ordering then the series is said to *converge uniformly on U* .

Proposition 1.2.22. (*Abel's Lemma*). Consider a multiple power series $\sum_{|J|=0}^{\infty} c_J(z-w)^J$ centred at the point $w \in \mathbb{C}^n$. Suppose that for some $\zeta \in \mathbb{C}^n$ there exists an $M \in \mathbb{R}$ such that $|c_J(\zeta-w)^J| \leq M$. Let $\rho = (\rho_1, \dots, \rho_n)$, where $\rho_k := |\zeta_k - w_k|$ for each $1 \leq k \leq n$. Then $\sum_{|J|=0}^{\infty} c_J(z-w)^J$ converges absolutely on the polydisk $\Delta^{(n)}(w, \rho)$ and uniformly on every compact set $K \subset \Delta^{(n)}(w, \rho)$.

PROOF. Fix an ordering of the multi-indices of the series and assume that $\rho_k \neq 0$ for each $1 \leq k \leq n$, otherwise $\Delta^{(n)}(w, \rho) = \emptyset$ and the claim is trivial. Further, for each $1 \leq k \leq n$, define a function $r_k : \Delta^{(n)}(w, \rho) \rightarrow \mathbb{C}$, $r_k(z) := \frac{1}{\rho_k} |z_k - w_k|$. For all $z \in \Delta^{(n)}(w, \rho)$, $0 \leq |r_k(z)| < 1$. Set $r(z) := (r_1(z), \dots, r_n(z))$ and suppose that $|c_J(\zeta-w)^J| \leq M$ for some $M \in \mathbb{R}$. Then

$$|c_J(z-w)^J| \leq |c_J| |r(z)|^J \rho^J \leq M |r(z)|^J < \infty, \quad (1)$$

where the last inequality follows from the fact that $|r(z)| < 1$ for all $z \in \Delta^{(n)}(w, \rho)$. This proves the absolute convergence of $\sum_{|J|=0}^{\infty} c_J(z-w)^J$ on $\Delta^{(n)}(w, \rho)$. Now let $K \subset \Delta^{(n)}(w, \rho)$ be a compact set. For each $1 \leq k \leq n$, set $R_k := \max_{z \in K} |r_k(z)| < 1$, and $R := (R_1, \dots, R_n)$. Then for each $z \in K$, $J \in \mathbb{N}_0^n$, the estimate (1) implies that $|c_J(z-w)^J| \leq MR^J$. By the Weierstrass M -test (see, e.g., [32, Corollary 11.4]) the series converges uniformly on K , independent of the choice of ordering. This completes the proof. \square

Corollary 1.2.23. Let G be the domain of convergence of the multiple power series $\sum_{|J|=0}^{\infty} c_J(z-w)^J$ centred at the point $w \in \mathbb{C}^n$. For any point $\zeta \in G$ we may find a polyradius $r = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$ such that $\zeta \in \Delta^{(n)}(w, r)$ and $\overline{\Delta}^{(n)}(w, r) \subset G$. Moreover, the convergence of $\sum_{|J|=0}^{\infty} c_J(z-w)^J$ is uniformly on $\overline{\Delta}^{(n)}(w, r)$.

PROOF. As in *Proposition 1.2.22*, we fix an ordering of the multi-indices. The domain of convergence G of $\sum_{|J|=0}^{\infty} c_J(z-w)^J$ is open, so we may choose points $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in G$ such that

$$|\alpha_k - w_k| > |\beta_k - w_k| > |\zeta_k - w_k|, \quad \forall 1 \leq k \leq n. \quad (2)$$

Set $\rho_k := |\alpha_k - w_k|$, $r_k := |\beta_k - w_k|$, where $\rho = (\rho_1, \dots, \rho_n)$ and $r = (r_1, \dots, r_n)$ the induced polyradii. From (2) it is clear that $\zeta \in \Delta^{(n)}(w, r) \subset \overline{\Delta}^{(n)}(w, r) \subset \Delta^{(n)}(w, \rho)$. Further, since the point α lies in the domain of convergence G , the series $\sum_{|J|=0}^{\infty} c_J(\alpha-w)^J$ converges. In particular, the terms of the series are uniformly bounded, i.e., there exists some $M \in \mathbb{R}$ such that $|c_J(\alpha-w)^J| \leq M$. Then from *Proposition 1.2.22* and (2), $\Delta^{(n)}(w, \rho) \subset G$, and so $\Delta^{(n)}(w, r) \subset G$. Since each r_k is finite, $\overline{\Delta}^{(n)}(w, r) \subset G$ is compact and *Proposition 1.2.22* informs us that the series $\sum_{|J|=0}^{\infty} c_J(z-w)^J$ converges uniformly on $\overline{\Delta}^{(n)}(w, r)$. \square

Remark 1.2.24. By *Proposition 1.2.22* the convergence of a multiple power series is independent of the ordering of the multi-indices. We, therefore, fix the ordering to be lexicographic and omit any further discussion of the ordering of the multi-indices. In particular, by the *domain of convergence (or a multiple power series)* we mean the domain of convergence relative to this lexicographic ordering.

Notational Remark 1.2.25. We fix the following notation. We denote the set of all convergent power series centred at the origin in \mathbb{C}^n by $\mathbb{C}\{z\}$, where $z = (z_1, \dots, z_n)$. The maximal ideal in $\mathbb{C}\{z\}$ is denoted by $\mathfrak{m}_{(z)}$. Similarly, for $w = (w_1, \dots, w_m)$ we write $\mathfrak{m}_{(w)}$ for the maximal ideal in $\mathbb{C}\{w\}$.

Remark 1.2.26. One may show that $\mathbb{C}\{z\}$ is a Noetherian local ring whose maximal ideal is given by

$$\mathfrak{m}_{(z)} = \langle z_1, \dots, z_n \rangle = \{f \in \mathbb{C}\{z\} : f(0) = 0\}$$

(see, e.g., [15, p. 81]).

We have the following important results for Noetherian local rings:

Theorem 1.2.27. Suppose that R is a Noetherian local ring with maximal ideal \mathfrak{m} .

- (i) (*Nakayama's Lemma*, [1, p. 22]). Suppose that M is a finitely generated R -module. Then the elements $x_1, \dots, x_p \in M$ generate the R -module M if and only if their equivalence classes $\bar{x}_1, \dots, \bar{x}_p \in M/\mathfrak{m}M$ generate the R/\mathfrak{m} -vector space $M/\mathfrak{m}M$.
- (ii) (*Krull's intersection theorem*, [15, p. 93]). Then

$$\bigcap_{k \in \mathbb{N}} \mathfrak{m}^k = \{0\},$$

$$\text{where } \mathfrak{m}^k = \underbrace{\mathfrak{m} \cdot \dots \cdot \mathfrak{m}}_{k \text{ times}}.$$

Definition 1.2.28. A complex vector space R is called a \mathbb{C} -algebra if there is a multiplication \cdot on R ,

$$\cdot : R \times R \longrightarrow R, \quad (x, y) \mapsto x \cdot y,$$

such that, together with vector addition, R is endowed with a ring structure (in the sense of *Definition 1.1.7*). Moreover, this ring and scalar multiplication satisfy:

$$\lambda(x \cdot y) = (\lambda x) \cdot y = x \cdot (\lambda y), \quad \forall \lambda \in \mathbb{C}, \quad \forall x, y \in R.$$

If R and S are \mathbb{C} -algebras, a *morphism of \mathbb{C} -algebras* is a map $\varphi : R \longrightarrow S$ such that

- (i) $\varphi(x + y) = \varphi(x) + \varphi(y)$, for all $x, y \in R$.
- (ii) $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$, for all $x, y \in R$.
- (iii) $\varphi(\lambda \cdot 1_R) = \lambda \cdot 1_S$, for all $\lambda \in \mathbb{C}$.

An *isomorphism of \mathbb{C} -algebras* is a morphism of \mathbb{C} -algebras which is bijective.

Definition 1.2.29. A \mathbb{C} -algebra R that is local as a ring is called a *local \mathbb{C} -algebra* if the composition of the canonical mappings

$$\mathbb{C} \cdot 1_R \rightarrow R \twoheadrightarrow R/\mathfrak{m}$$

is an isomorphism of fields, i.e., the identification of \mathbb{C} with $\mathbb{C} \cdot 1_R$ determines a complex vector space isomorphism $R \cong \mathbb{C} \oplus \mathfrak{m}$.

Definition 1.2.30. We say that a \mathbb{C} -algebra R is an *analytic \mathbb{C} -algebra* if R is isomorphic (as a \mathbb{C} -algebra) to $\mathbb{C}\{z\}/I$ for some $n \geq 0$ and ideal $I \subset \mathbb{C}\{z\}$. A *morphism of analytic \mathbb{C} -algebras* is a morphism of \mathbb{C} -algebras in the sense of *Definition 1.2.28*.

Lemma 1.2.31. A morphism $f : \mathbb{C}\{z\} \rightarrow \mathbb{C}\{w\}$ of analytic \mathbb{C} -algebras is local, i.e., f maps the maximal ideal in $\mathbb{C}\{z\}$ into the maximal ideal in $\mathbb{C}\{w\}$.

PROOF. To see that $f(\mathfrak{m}_{(z)}) \subset \mathfrak{m}_{(w)}$, observe that for $x \in \mathfrak{m}_{(z)}$, $f(x) = y + \lambda$ for some $y \in \mathfrak{m}_{(w)}$, $\lambda \in \mathbb{C}$. We want to show that $\lambda = 0$. Indeed, if λ is non-zero, then $y - \lambda$ is invertible, and $f(x) - \lambda = f(x - \lambda) = y$ is invertible also. Since $y \in \mathfrak{m}_{(w)}$, it follows that y is not invertible, and so λ must be zero, as required. \square

In the interests of brevity, we omit the proof of the following lemma. The proof is a simple application of (ii) of *Theorem 1.2.27* and may be found in [37, p. 69].

Lemma 1.2.32. Let $\varphi : \mathbb{C}\{z\} \rightarrow \mathbb{C}\{w\}$ be a morphism of \mathbb{C} -algebras, and suppose that $(f_k)_{k \in \mathbb{N}}$ is a summable sequence of power series in $\mathbb{C}\{z\}$. Then $(\varphi(f_k))_{k \in \mathbb{N}}$ is a summable sequence of power series in $\mathbb{C}\{w\}$, and $\varphi(\sum_{k \in \mathbb{N}} f_k) = \sum_{k \in \mathbb{N}} \varphi(f_k)$.

Note that *Definition 1.2.16* and *Lemma 1.2.20* transfers to power series in $\mathbb{C}\{z\}$ without change.

Proposition 1.2.33. Let $g_1, \dots, g_n \in \mathfrak{m}_{(w)}$. There is a unique morphism of \mathbb{C} -algebras $\Phi : \mathbb{C}\{z\} \rightarrow \mathbb{C}\{w\}$, such that $\Phi(z_k) = g_k$ for each $1 \leq k \leq n$.

PROOF. Let us first show that Φ exists. To this end, let $f(z) = \sum_{|J|=0}^{\infty} c_J z^J$ be a convergent power series in $\mathbb{C}\{z\}$. We claim that $\Phi(f) := \sum_{|J|=0}^{\infty} c_J g^J$ is the desired map. We need to show that $\Phi(f)$ is a convergent power series in $\mathbb{C}\{w\}$. Choose $\delta = (\delta_1, \dots, \delta_n) \in (\mathbb{R}_{>0})^n$ such that $\|f\|_{\delta} < \infty$, and observe that since each g_k lies in $\mathfrak{m}_{(w)}$, and therefore $g_k(0) = 0$ for each k . By (iii) of *Lemma 1.2.20*, there is some $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{R}_{>0})^n$ such that $\|g_k\|_{\varepsilon} < \delta_k$ for each k . We write $f(z) = \sum_{\ell=0}^{\infty} p_{\ell}$ as a series of homogeneous polynomials. Then, by (i) and (ii) of *Lemma 1.2.20*,

$$\begin{aligned} \|\Phi(f)\|_{\varepsilon} &= \left\| \sum_{\ell=0}^{\infty} \Phi(p_{\ell}) \right\|_{\varepsilon} \\ &\leq \sum_{\ell=0}^{\infty} \|\Phi(p_{\ell})\|_{\varepsilon} \\ &= \sum_{\ell=0}^{\infty} \left\| \sum_{|J|=\ell} c_{j_1 \dots j_n} \Phi(z_1)^{j_1} \dots \Phi(z_n)^{j_n} \right\|_{\varepsilon} \\ &\leq \sum_{|J|=0}^{\infty} |c_{j_1 \dots j_n}| \cdot \|\Phi(z_1)\|_{\varepsilon}^{j_1} \dots \|\Phi(z_n)\|_{\varepsilon}^{j_n} \\ &\leq \sum_{|J|=0}^{\infty} |c_J| \cdot \delta^J = \|f\|_{\delta} < \infty. \end{aligned}$$

So $\Phi(f)$ lies in $\mathbb{C}\{w\}$, and it is clear that Φ defines a morphism of \mathbb{C} -algebras.

To show that Φ is unique, suppose $\Psi : \mathbb{C}\{z\} \rightarrow \mathbb{C}\{w\}$ is another morphism of \mathbb{C} -algebras such that $\Psi(z_k) = g_k$ for each $1 \leq k \leq n$. Then Φ and Ψ agree on the polynomial subring $\mathbb{C}[z]$. Hence, for each $f \in \mathbb{C}\{z\}$, and each

$k \in \mathbb{N}$, we have $(\Phi - \Psi)(f) \in \mathfrak{m}_{(w)}^k$. Therefore, by (ii) of *Theorem 1.2.27*,

$$(\Phi - \Psi)(f) \in \bigcap_{k \in \mathbb{N}} \mathfrak{m}_{(w)}^k = \{0\}.$$

□

Now equipped with an understanding of multiple power series, we conclude this section by mentioning the power series representation of holomorphic functions. This representation is achieved using *Corollary 1.2.11* in an analogous manner to the one-variable case.

Theorem 1.2.34. Suppose that $f : \overline{\Delta}^{(n)}(w, r) \rightarrow \mathbb{C}$ is holomorphic on $\Delta^{(n)}(w, r)$ and continuous on $\overline{\Delta}^{(n)}(w, r)$, then for all $z \in \Delta^{(n)}(w, r)$,

$$f(z) = \sum_{|J|=0}^{\infty} c_J (z - w)^J, \quad c_J = \frac{1}{J!} \left. \frac{\partial^{|J|} f}{\partial z^J} \right|_w,$$

where $\left. \frac{\partial^{|J|} f}{\partial z^J} \right|_w := \left. \frac{\partial^{|J|} f}{\partial z_1^{j_1} \cdots \partial z_n^{j_n}} \right|_w$.

Convention 1.2.35. In light of *Theorem 1.2.34*, we may use the expressions *holomorphic* and *analytic* interchangeably.

§1.3. THE CALCULUS OF DIFFERENTIAL FORMS

In this section we develop the calculus of differential forms which will be the primary source of machinery in the proof of Hartogs' extension theorem. Differential forms will also play a central role in Dolbeault cohomology which is treated in Chapter 3.2, (see also, e.g., [18], [26, Chapter 1.D], [31, Chapter 1.3], [59, Chapter 2.6]).

We remind the reader that a smooth manifold M is studied by means of its *tangent bundle* TM , and its k -*form bundle* $\Lambda^k T^*M$. Here, T^*M denotes the *cotangent bundle*, the dual of the tangent bundle. We first discuss some elementary linear algebra that will be applied to the tangent spaces of a smooth manifold.

Throughout this section, V denotes a real vector space of dimension $2n$.

Definition 1.3.1. An *almost complex structure* on V is an endomorphism $\mathcal{J} : V \rightarrow V$ satisfying $\mathcal{J}^2 = -\text{Id}_V$.

Example 1.3.2. An almost complex structure on \mathbb{R}^2 is given by the matrix

$$\mathcal{J} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Any even-dimensional vector V may be equipped with an almost complex structure \mathcal{J} . Moreover, using \mathcal{J} , we may endow V with a complex scalar multiplication by setting $\lambda \cdot v := \text{Re}(\lambda)v + \mathcal{J}\text{Im}(\lambda)v$.

Definition 1.3.3. The complex vector space $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ is called the *complexification of V* .

There is a natural inclusion of V into $V^{\mathbb{C}}$ given by the map $v \mapsto v \otimes 1$, and we identify V with the subspace of $V^{\mathbb{C}}$ which is invariant under complex conjugation. Note that complex conjugation in $V^{\mathbb{C}}$ is defined by $(\overline{v \otimes \lambda}) := v \otimes \bar{\lambda}$, where $v \in V$, $\lambda \in \mathbb{C}$.

Notational Remark 1.3.4. We will abuse notation and denote the extension of $\mathcal{J} \in \text{End}(V)$ to $\text{End}(V^{\mathbb{C}})$ by \mathbb{C} -linearity is also denoted by \mathcal{J} .

Definition 1.3.5. The eigenvalues of $\mathcal{J} : V^{\mathbb{C}} \rightarrow V^{\mathbb{C}}$ are $\pm i$. The respective eigenspaces of J corresponding to i and $-i$ are

$$V^{1,0} = \{v \in V^{\mathbb{C}} : \mathcal{J}v = iv\}, \quad \text{and} \quad V^{0,1} = \{v \in V^{\mathbb{C}} : \mathcal{J}v = -iv\}.$$

Lemma 1.3.6. ([31, p. 26]) The complexification of V splits as the direct sum:

$$V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

Lemma 1.3.7. ([31, p. 26]) The dual vector space $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ has a natural almost complex structure given by $\mathcal{J}(f)(v) = f(\mathcal{J}(v))$. The induced decomposition on $(V^*)^{\mathbb{C}} = \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = (V^{\mathbb{C}})^*$ is given by

$$\begin{aligned} (V^*)^{1,0} &= \{f \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) : f(\mathcal{J}(v)) = if(v)\} = (V^{1,0})^* \\ (V^*)^{0,1} &= \{f \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) : f(\mathcal{J}(v)) = -if(v)\} = (V^{0,1})^* \end{aligned}$$

Definition 1.3.8. For $k \in \mathbb{N}_0$, the k th tensor power of V is the set

$$T^k V := V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}.$$

We adopt the convention that $T^0 V := \mathbb{R}$. If V is a complex vector space, then $T^0 V := \mathbb{C}$.

Definition 1.3.9. The *tensor algebra* $\mathcal{T}(V)$ of V is the direct sum

$$\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} T^k V,$$

with multiplication given by the canonical isomorphism $T^k V \otimes T^\ell V \cong T^{k+\ell} V$ induced from the tensor product.

Definition 1.3.10. Let I be the two-sided ideal generated by the elements of the form $v \otimes v$, where $v \in V$. The *exterior algebra* $\Lambda(V)$ of V is the quotient algebra

$$\Lambda(V) := \mathcal{T}(V)/I.$$

The *complexification* of the exterior algebra is given by $\Lambda_{\mathbb{C}}(V) := \Lambda(V) \otimes \mathbb{C}$.

One may readily verify that $I = \bigoplus_{k \in \mathbb{N}_0} (I \cap V^{\otimes k})$, and for each $k \in \mathbb{N}_0$ we set $\Lambda^k(V) = T^k V / I_k$, where $I_k := I \cap V^{\otimes k}$. We then define a canonical projection map $\pi : V^{\otimes k} \rightarrow \Lambda^k(V)$ by setting

$$\pi(v_1 \otimes \cdots \otimes v_k) := v_1 \wedge \cdots \wedge v_k,$$

where \wedge denotes the *wedge product*.

Lemma 1.3.11. The wedge product \wedge satisfies the following properties.

- (i) The wedge product is an *alternating product*, i.e., $v \wedge v = 0$ for all $v \in V$.
- (ii) The wedge product is anticommutative, i.e., for any $\sigma \in S_k$, where S_k denotes the symmetric group on k letters,

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \operatorname{sgn}(\sigma) v_1 \wedge \cdots \wedge v_k.$$

Now let M be a smooth $2n$ -dimensional manifold. Denote the local coordinates of M by $(x_1, \dots, x_n, y_1, \dots, y_n)$. For each $p \in M$, the tangent space $T_p M$ is a real $2n$ -dimensional vector space given by

$$T_p M := \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p, \frac{\partial}{\partial y_1} \Big|_p, \dots, \frac{\partial}{\partial y_n} \Big|_p \right\}.$$

We may endow each tangent space $T_p M$ with an almost structure \mathcal{J}_p by setting

$$\mathcal{J}_p \left(\frac{\partial}{\partial x_k} \Big|_p \right) := \frac{\partial}{\partial y_k} \Big|_p, \quad \text{and} \quad \mathcal{J}_p \left(\frac{\partial}{\partial y_k} \Big|_p \right) := -\frac{\partial}{\partial x_k} \Big|_p,$$

for each $1 \leq k \leq n$.

Definition 1.3.12. An *almost complex manifold* is a pair $M := (M, \mathcal{J})$, where M is a smooth manifold of dimension $2n$, and \mathcal{J} denotes the almost complex structure \mathcal{J}_p on each tangent space $T_p M$, i.e., for each $p \in M$, we have $\mathcal{J}_p^2 = -\operatorname{Id}_{T_p M}$.

Remark 1.3.13. Despite the fact that we are yet to define complex manifolds, let us note that the distinction between an almost complex manifold and a complex manifold is whether each of these locally-defined complex structures $\mathcal{J}_p \in \operatorname{End}(T_p M)$ may be glued together to obtain an endomorphism $\mathcal{J} \in \operatorname{End}(TM)$. The spheres \mathbb{S}^{4k} , $k \in \mathbb{N}$, are examples of almost complex manifolds which are not complex manifolds, (see, e.g., [2], [45, Chapter 7], [39, Chapter IX], [46, Chapter 8.7]).

As promised, we now apply the linear algebra we have discussed so far to the tangent spaces of smooth manifolds. We will assume throughout the remainder of this section that M is an almost complex manifold of dimension $2n$.

Definition 1.3.14. The *complexified tangent space* of M (at p) is the complex vector space

$$T_p^{\mathbb{C}} M := T_p M \otimes_{\mathbb{R}} \mathbb{C}.$$

By *Lemma 1.3.6* the almost complex structure \mathcal{J}_p induces the splitting

$$T_p^{\mathbb{C}} M = T_p^{1,0} M \oplus T_p^{0,1} M,$$

where $T_p^{1,0} M := \{v \in T_p^{\mathbb{C}} M : \mathcal{J}_p v = iv\}$ and $T_p^{0,1} M := \{v \in T_p^{\mathbb{C}} M : \mathcal{J}_p v = -iv\}$. The vector spaces $T_p^{1,0} M$ and $T_p^{0,1} M$ are commonly referred to as the *holomorphic* and *anti-holomorphic tangent spaces* of M at p , respectively.

The *complexified tangent bundle* $\pi : T^{\mathbb{C}}M \rightarrow M$ is the complexification of the tangent bundle, i.e., $T^{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$. In other words, the fibres of $\pi : T^{\mathbb{C}}M \rightarrow M$ are given by $\pi^{-1}(p) = T_p M \otimes_{\mathbb{R}} \mathbb{C}$ for each $p \in M$. The above decomposition on each tangent space induces the splitting of $T^{\mathbb{C}}M$ into the direct sum of the *holomorphic tangent bundle* $T^{1,0}M$ and the *anti-holomorphic tangent bundle* $T^{0,1}M$. Moreover, $T^{1,0}M$ and $T^{0,1}M$ are respectively trivialised (in a neighbourhood of p) by

$$\frac{\partial}{\partial z_k} := \frac{1}{2} \left(\frac{\partial}{\partial x_k} - \mathcal{J}_p \frac{\partial}{\partial y_k} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_k} := \frac{1}{2} \left(\frac{\partial}{\partial x_k} + \mathcal{J}_p \frac{\partial}{\partial y_k} \right),$$

where $z_k = x_k + \mathcal{J}_p y_k$, for each $k = 1, \dots, n$.

Definition 1.3.15. The *complexified exterior algebra* (at $p \in M$) is the \mathbb{C} -algebra

$$\Lambda_{\mathbb{C}}^{\bullet}(T_p^* M) := \bigoplus_{k=0}^{\infty} \Lambda^k(T_p^* M) \otimes_{\mathbb{R}} \mathbb{C}.$$

The set of all $\Lambda_{\mathbb{C}}^{\bullet}(T_p^* M)$ is called the *complexified exterior bundle*, and for any $k \in \mathbb{N}$, we call $\Lambda_{\mathbb{C}}^k M$ the *complexified k -form bundle*.

We have the following two subbundles of $\Lambda_{\mathbb{C}}^1 M$:

$$\begin{aligned} \Lambda^{1,0} M &:= \{ \xi \in \Lambda_{\mathbb{C}}^1 M : \xi(v) = 0, \quad \forall v \in T^{0,1} M \}, \\ \Lambda^{0,1} M &:= \{ \xi \in \Lambda_{\mathbb{C}}^1 M : \xi(v) = 0, \quad \forall v \in T^{1,0} M \}. \end{aligned}$$

The k th exterior power of $\Lambda^{1,0} M$ and $\Lambda^{0,1}$ is respectively denoted by $\Lambda^{k,0} M$ and $\Lambda^{0,k} M$. Further, we have the splitting

$$\Lambda_{\mathbb{C}}^k M = \bigoplus_{p+q=k} \Lambda^{p,q} M.$$

Definition 1.3.16. For an open set $U \subseteq M$, we denote by $\Omega^k(U)$ the space of smooth sections of $\Lambda^k M$ over U . Elements of $\Omega^k(U)$ are referred to as *smooth k -forms* (or *differential k -forms*) on U . Similarly, we denote by $\Omega^{p,q}(U)$ the space of smooth sections of $\Lambda^{p,q} M$ over U . Elements of $\Omega^{p,q}(U)$ are referred to as *smooth (p, q) -forms* (or *differential (p, q) -forms*) on U .

Remark 1.3.17. For our purposes, we will only be concerned only with forms defined on domains in \mathbb{C}^n . In this case, the almost complex structure \mathcal{J} is inherited from the multiplication by i defined on \mathbb{C}^n . Moreover, in the coordinates $z = (z_1, \dots, z_n)$, a smooth (p, q) -form $\omega \in \Omega^{p,q}(G)$, may be written

$$\omega = \sum'_{I,J} \omega_{IJ} dz_I \wedge d\bar{z}_J,$$

where the functions ω_{IJ} are smooth in G , $dz_I := dz_{i_1} \wedge \dots \wedge dz_{i_p}$, and $d\bar{z}_J := d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$. Here, the prime summation signifies that the summation is ordered, i.e., $1 \leq i_1 < \dots < i_p \leq n$.

Note that if the functions ω_{IJ} may be taken to be holomorphic on U , we say that $\omega = \sum_{I,J} \omega_{IJ} dz_I \wedge d\bar{z}_J$ is a *holomorphic (p, q) -form*.

Remark 1.3.18. We remind ourselves that on the space $\Omega^k(G)$ of k -forms we have a differential operator given by the *exterior derivative* $d : \Omega^k(G) \rightarrow \Omega^{k+1}(G)$, specified by the formula

$$d \left(\sum_I f_I dx_I \right) = \sum_{I,k} \frac{\partial f_I}{\partial x_k} dx_k \wedge dx_I.$$

For domains $G \subseteq \mathbb{C}^n$ we have a splitting of k -forms given by

$$\Omega^k(G) = \Omega^{p,0}(G) \oplus \Omega^{0,q}(G),$$

where $p+q=k$. This induces a splitting of the exterior derivative $d = \partial + \bar{\partial}$, where $\partial : \Omega^{p,0}(G) \rightarrow \Omega^{p+1,0}(G)$ and $\bar{\partial} : \Omega^{0,q}(G) \rightarrow \Omega^{0,q+1}(G)$ are defined by

$$\partial \left(\sum_I f_I dz_I \right) = \sum_{I,k} \frac{\partial f_I}{\partial z_k} dz_k \wedge dz_I, \quad \text{and} \quad \bar{\partial} \left(\sum_J f_J d\bar{z}_J \right) = \sum_{J,k} \frac{\partial f_J}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_J.$$

We will be concerned primarily with the *Dolbeault operator* $\bar{\partial}$. We extend $\bar{\partial}$ to be defined on $\Omega^{p,q}(G)$ for possibly non-zero p by setting

$$\bar{\partial} \left(\sum_{I,J} f_I dz_I \wedge d\bar{z}_J \right) = \sum_{I,J,k} \frac{\partial f_I}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J.$$

The following proposition details some elementary properties of the Dolbeault operator $\bar{\partial}$.

Proposition 1.3.19. The Dolbeault operator $\bar{\partial} : \Omega^{p,q}(G) \rightarrow \Omega^{p,q+1}(G)$, satisfies the following properties.

- (i) (*Invariance*). $\bar{\partial}$ is well-defined, independent of the choice of coordinates.
- (ii) (*Linearity*). For all $\omega, \eta \in \Omega^{p,q}(G)$, $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\bar{\partial}(\lambda_1\omega + \lambda_2\eta) = \lambda_1\bar{\partial}\omega + \lambda_2\bar{\partial}\eta.$$

- (iii) (*Leibniz Rule*). If $\omega \in \Omega^{p,q}(G)$ and $\eta \in \Omega^{r,s}(G)$, then

$$\bar{\partial}(\omega \wedge \eta) = \bar{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \bar{\partial}\eta.$$

- (iv) (*Nilpotence*). $\bar{\partial} \circ \bar{\partial} = 0$. More explicitly, by $\bar{\partial} \circ \bar{\partial}$ we mean the composition of $\bar{\partial} : \Omega^{p,q}(G) \rightarrow \Omega^{p,q+1}(G)$ and $\bar{\partial} : \Omega^{p,q+1}(G) \rightarrow \Omega^{p,q+2}(G)$.

PROOF. Statements (i) and (ii) are clear. Let $\omega = \sum_{I,J} \omega_{IJ} dz_I \wedge d\bar{z}_J$ and $\eta = \sum_{K,L} \eta_{KL} dz_K \wedge d\bar{z}_L$. To prove (iii) we need to simply observe that the $\bar{\partial}$ operator on smooth functions, i.e., $(0,0)$ -forms, is given by

$$f \mapsto \sum_{\alpha=1}^n \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha.$$

Therefore, from the usual Leibniz rule for functions, we have

$$\begin{aligned}
\bar{\partial}(\omega \wedge \eta) &= \bar{\partial} \left(\sum_{I,J,K,L} \omega_{IJ} \eta_{KL} dz_I \wedge d\bar{z}_J \wedge dz_K \wedge d\bar{z}_L \right) \\
&= \sum_{I,J,K,L,\alpha} \left(\eta_{KL} \frac{\partial \omega_{IJ}}{\partial \bar{z}_\alpha} d\bar{z}_\alpha + \omega_{IJ} \frac{\partial \eta_{KL}}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \right) \wedge dz_I \wedge d\bar{z}_J \wedge dz_K \wedge d\bar{z}_L \\
&= \left(\sum_{I,J,\alpha} \frac{\partial \omega_{IJ}}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge dz_I \wedge d\bar{z}_J \right) \wedge \left(\sum_{K,L} \eta_{KL} dz_K \wedge d\bar{z}_L \right) \\
&\quad + \sum_{I,J,K,L,\alpha} \omega_{IJ} \frac{\partial \eta_{KL}}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge dz_I \wedge d\bar{z}_J \wedge dz_K \wedge d\bar{z}_L \\
&= \bar{\partial}\omega \wedge \eta + \left(\sum_{I,J} \omega_{IJ} dz_I \wedge d\bar{z}_J \right) (-1)^{|I|+|J|} \sum_{K,L,\alpha} \frac{\partial \eta_{KL}}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge dz_K \wedge d\bar{z}_L \\
&= \bar{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \bar{\partial}\eta.
\end{aligned}$$

To prove (iv), we have the similar computation:

$$\begin{aligned}
\bar{\partial}(\bar{\partial}\omega) &= \bar{\partial} \left(\sum_{I,J,\alpha} \frac{\partial \omega_{IJ}}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge dz_I \wedge d\bar{z}_J \right) \\
&= \sum_{I,J,\alpha,\beta} \frac{\partial^2 \omega_{IJ}}{\partial \bar{z}_\beta \partial \bar{z}_\alpha} d\bar{z}_\beta \wedge d\bar{z}_\alpha \wedge dz_I \wedge d\bar{z}_J = 0,
\end{aligned}$$

where the last equality follows from the symmetry of the complex Hessian $\left(\frac{\partial^2 \omega_{IJ}}{\partial \bar{z}_\beta \partial \bar{z}_\alpha} \right)$ and the anti-symmetry of the wedge product¹. \square

There is a well-established integration theory for differential forms on smooth manifolds. This theory transfers over to complex manifolds with effectively no change (see, e.g., [40, Chapter 14], [59, Chapter 2]. for further details).

Theorem 1.3.20. (*Stokes' theorem*). Let $G \subset \mathbb{C}^n$ be a bounded domain with smooth boundary ∂G . Then for any smooth $(n-1)$ -form $\omega \in \Omega^{n-1}(U)$, where U is a neighbourhood of the boundary ∂G ,

$$\int_G d\omega = \int_{\partial G} \omega.$$

From *Theorem 1.3.20*, we obtain a generalisation of the standard Cauchy integral formula in one complex variable.

¹It is difficult to write the details down explicitly without the notation becoming too cumbersome.

Proposition 1.3.21. (*Cauchy–Pompeiu formula* [54]). Let $G \subset \mathbb{C}$ be a bounded domain with smooth boundary. Let U be an open neighbourhood of \overline{G} and suppose that f is an \mathbb{R} -differentiable function on U . Then for all $z \in G$,

$$f(z) = -\frac{1}{2\pi i} \int_{\partial G} \frac{f(\zeta)}{\zeta - z} d\bar{\zeta} + \frac{1}{2\pi i} \int_G \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}. \quad (3)$$

PROOF. Fix a point $z \in G$ and let $\Delta(z, r) \subset G$ be a sufficiently small disk centred at z such that $\overline{\Delta}(z, r) \subset G$. We will denote the boundary of $\Delta(z, r)$ by Γ_r and set $G_r := G \setminus \Delta(z, r)$. Let us write

$$\frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} = \frac{\partial}{\partial \bar{\zeta}} \left(\frac{f(\zeta)}{\zeta - z} \right) d\bar{\zeta} \wedge d\zeta = d \left(\frac{f(\zeta)}{\zeta - z} d\zeta \right),$$

where in the last equality we have used the fact that $d\zeta \wedge d\zeta = 0$. Then by *Theorem 1.3.20*,

$$\int_{G_r} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} = \int_{G_r} d \left(\frac{f(\zeta)}{\zeta - z} d\zeta \right) = \int_{\partial G} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\Gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta. \quad (4)$$

We want to take the limit of (4) as $r \rightarrow 0$. We consider first the integral

$$\int_{G_r} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} = \int_G \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} - \int_{\Delta(z, r)} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}. \quad (5)$$

The integral over G in (5) is independent of r , so we need only concern ourselves with the integral over the disk $\Delta(z, r)$. Since $\frac{\partial f}{\partial \bar{\zeta}}$ is continuous and $\frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}$ is a bounded measure, we see that

$$\int_{\Delta(z, r)} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} \rightarrow 0$$

as $r \rightarrow 0$. Proceeding in a similar manner for the integral over Γ_r in (4), write

$$\int_{\Gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_0^{2\pi} f(z + re^{i\vartheta}) id\vartheta.$$

Then

$$\begin{aligned} \left| \int_0^{2\pi} f(z + re^{i\vartheta}) id\vartheta - 2\pi i f(z) \right| &= \left| \int_0^{2\pi} f(z + re^{i\vartheta}) id\vartheta - i \int_0^{2\pi} f(z) d\vartheta \right| \\ &\leq \int_0^{2\pi} |f(z + re^{i\vartheta}) - f(z)| d\vartheta \\ &\leq 2\pi r \max_{0 \leq \vartheta \leq 2\pi} |f(z + re^{i\vartheta})| \rightarrow 0 \end{aligned}$$

as $r \rightarrow 0$. Hence, by taking the limit as $r \rightarrow 0$ of (4),

$$\int_G \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} = \int_{\partial G} \frac{f(\zeta)}{\zeta - z} d\zeta - 2\pi i f(z). \quad (6)$$

Rearranging (6) and using the fact that $d\bar{\zeta} \wedge d\zeta = -d\zeta \wedge d\bar{\zeta}$ we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_G \frac{\partial f}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}. \quad (7)$$

Conjugating this expression, we see that

$$\begin{aligned}
\overline{f(z)} &= -\frac{1}{2\pi i} \int_{\partial G} \frac{\overline{f(\zeta)}}{\zeta - z} d\bar{\zeta} - \frac{1}{2\pi i} \int_G \frac{\overline{\partial f}}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} \\
&= -\frac{1}{2\pi i} \int_{\partial G} \frac{\overline{f(\zeta)}}{\bar{\zeta} - \bar{z}} d\bar{\zeta} - \frac{1}{2\pi i} \int_G \frac{\overline{\partial f}}{\partial \zeta} \frac{d\bar{\zeta} \wedge d\zeta}{\bar{\zeta} - \bar{z}} \\
&= -\frac{1}{2\pi i} \int_{\partial G} \frac{\overline{f(\zeta)}}{\bar{\zeta} - \bar{z}} d\bar{\zeta} + \frac{1}{2\pi i} \int_G \frac{\overline{\partial f}}{\partial \zeta} \frac{d\zeta \wedge d\bar{\zeta}}{\bar{\zeta} - \bar{z}},
\end{aligned}$$

which is exactly (3) with f replaced by \bar{f} . □

Remark 1.3.22. Observe that if f is holomorphic in U , we recover the familiar Cauchy integral of one complex variable from (7).

Proposition 1.3.23. Let $G \subset \mathbb{C}$ be a bounded domain with smooth boundary. Let U be an open neighbourhood of \bar{G} with $f \in \mathcal{C}^1(U, \mathbb{C})$, as in *Proposition 1.3.21*. Then there exists a function $g \in \mathcal{C}^1(U, \mathbb{C})$ such that

$$\frac{\partial g}{\partial \bar{z}}(z) = f(z).$$

PROOF. The proof is analogous to the proof of *Proposition 1.3.21*. Fix a point $z \in G$ and let $\Delta(z, r) \subset G$ be a sufficiently small disk centred at z such that $\bar{\Delta}(z, r) \subset G$. As before, let Γ_r denote the boundary of $\Delta(z, r)$ and set $G_r := G \setminus \Delta(z, r)$. Now for all $\zeta \neq z$,

$$d \log |\zeta - z|^2 = \frac{d\zeta}{\zeta - z} + \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}}. \quad (8)$$

Then by *Theorem 1.3.20*,

$$\begin{aligned}
&\int_{\partial G} f(\zeta) \log |\zeta - z|^2 d\bar{\zeta} - \int_{\Gamma_r} f(\zeta) \log |\zeta - z|^2 d\bar{\zeta} \\
&= \int_{G_r} d \left(f(\zeta) \log |\zeta - z|^2 d\bar{\zeta} \right) \\
&= \int_{G_r} \frac{\partial f}{\partial \zeta}(\zeta) \log |\zeta - z|^2 d\zeta \wedge d\bar{\zeta} + \int_{G_r} f(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}. \quad (9)
\end{aligned}$$

To determine the limit of the above equation as $r \rightarrow 0$, set $\zeta = z + re^{i\vartheta}$. Then $d\zeta = ire^{i\vartheta} d\vartheta$ and $d\bar{\zeta} = -ire^{-i\vartheta} d\vartheta$. Hence,

$$\begin{aligned}
\left| \int_{\Gamma_r} f(\zeta) \log |\zeta - z|^2 d\bar{\zeta} \right| &= \left| \int_0^{2\pi} f(z + re^{i\vartheta}) \log |re^{i\vartheta}|^2 (-ire^{-i\vartheta}) d\vartheta \right| \\
&\leq \left| 2 \int_0^{2\pi} f(z + re^{i\vartheta}) r^2 d\vartheta \right| \\
&\leq 4\pi r^2 \max_{0 \leq \vartheta \leq 2\pi} |f(z + re^{i\vartheta})| \rightarrow 0
\end{aligned}$$

as $r \rightarrow 0$. Therefore, using (9), we have

$$\int_{\partial G} f(\zeta) \log |\zeta - z|^2 d\bar{\zeta} = \int_G \frac{\partial f}{\partial \zeta}(\zeta) \log |\zeta - z|^2 d\zeta \wedge d\bar{\zeta} + \int_G f(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

Define

$$g(z) := \frac{1}{2\pi i} \int_G f(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

Then

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \int_G f(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_{\partial G} f(\zeta) \log |\zeta - z|^2 d\bar{\zeta} - \frac{1}{2\pi i} \int_G \frac{\partial f}{\partial \zeta}(\zeta) \log |\zeta - z|^2 d\zeta \wedge d\bar{\zeta}. \end{aligned} \quad (10)$$

We claim that $\frac{\partial g}{\partial \bar{z}} = f$. Differentiating (10), we see from (8) that

$$\begin{aligned} \frac{\partial g}{\partial \bar{z}} &= \frac{1}{2\pi i} \int_{\partial G} f(\zeta) \left(\frac{\partial}{\partial \bar{z}} \log |\zeta - z|^2 \right) d\bar{\zeta} - \frac{1}{2\pi i} \int_G \frac{\partial f}{\partial \zeta}(\zeta) \left(\frac{\partial}{\partial \bar{z}} \log |\zeta - z|^2 \right) d\zeta \wedge d\bar{\zeta} \\ &= -\frac{1}{2\pi i} \int_{\partial G} \frac{f(\zeta)}{\bar{\zeta} - \bar{z}} d\bar{\zeta} + \frac{1}{2\pi i} \int_G \frac{\partial f}{\partial \zeta}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\bar{\zeta} - \bar{z}}, \end{aligned} \quad (11)$$

where differentiation under the integral sign is justified since the resulting integrand is still integrable (see, e.g., [35, p. 213]). Since this is exactly the Cauchy–Pompeiu representation formula (3), this completes the proof. \square

§1.4. HARTOGS' EXTENSION THEOREM

It was known even in the time of Riemann [57] and Weierstrass [64] that a function holomorphic in a domain $G \setminus \{p\} \subseteq \mathbb{C}$ and locally bounded in a neighbourhood of p , extends to a function holomorphic over all of G . In this section, we make precise what it means for a holomorphic function to extend analytically and prove the extension theorem due to F. Hartogs [27].

Definition 1.4.1. Let $G \subset \mathbb{C}^n$ be an open set, $f \in \mathcal{O}(G)$, and $p \in \mathbb{C}^n \setminus G$. We say that f *extends analytically to p* if there is a connected open neighbourhood U of p and an analytic function $\tilde{f} \in \mathcal{O}(U)$ such that, for some non-empty open set $V \subset G \cap U$, $\tilde{f}|_V = f|_V$.

Remark 1.4.2. Notice that since we do not require f and \tilde{f} to coincide on all of $G \cap U$, it may not necessarily be the case that \tilde{f} is analytic on $G \cup U$. Moreover, *Definition 1.4.1* ensures that the analytic extension \tilde{f} is single-valued.

Theorem 1.4.3. (*Hartogs' Extension Theorem*). Let $G \subset \mathbb{C}^{n>1}$ be a domain, not necessarily bounded. Let $K \subset G$ be a compact set with $G \setminus K$ connected. If $f \in \mathcal{O}(G \setminus K)$, then there exists a function $\tilde{f} \in \mathcal{O}(G)$ such that $\tilde{f}|_{G \setminus K} = f$. In other words, every function analytic on $G \setminus K$ extends analytically to a function analytic on all of G .

PROOF. Let U be an open neighbourhood of K such that $\bar{U} \subset G$. Take $\rho \in \mathcal{C}_0^\infty(G)$ to be a smooth function compactly supported² in G , with $\rho(z) = 1$ for all $z \in U$. For each $1 \leq k \leq n$ we will define

²We remind the reader that the support of the function ρ is the set $\text{supp}(\rho) := \overline{\{z \in G : \rho(z) \neq 0\}}$. The space of smooth functions of compact support in G is denoted by $\mathcal{C}_0^\infty(G)$.

$g_k : G \longrightarrow \mathbb{C}$ by

$$g_k := f \cdot \frac{\partial \rho}{\partial \bar{z}_k}.$$

Note that since $\rho(z) = 1$ for all $z \in U$, $g_k(z) = 0$ for all $z \in U$. Therefore, g only takes non-zero values in $\text{supp}(\rho) \cap G \setminus U$ and in $G \setminus U$, f is defined and holomorphic.

Proceeding formally,

$$\frac{\partial g_k}{\partial \bar{z}_\ell} = \frac{\partial f}{\partial \bar{z}_\ell} \frac{\partial \rho}{\partial \bar{z}_k} + f \cdot \frac{\partial^2 \rho}{\partial \bar{z}_k \partial \bar{z}_\ell} = f \cdot \frac{\partial^2 \rho}{\partial \bar{z}_k \partial \bar{z}_\ell} = \frac{\partial g_\ell}{\partial \bar{z}_k}. \quad (12)$$

Let us now fix $z' = (z_2, \dots, z_n) \in \mathbb{C}^{n-1}$ and define

$$u(z_1, z') := \frac{1}{2\pi i} \int_G \frac{g_1(\zeta, z')}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}.$$

By *Proposition 1.3.23*, we see that $\frac{\partial u}{\partial \bar{z}_1} = g_1$, and for all $k > 1$,

$$\begin{aligned} \frac{\partial u}{\partial \bar{z}_k} &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_1}{\partial \bar{z}_k}(\zeta, z') \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z_1} \\ &\stackrel{(12)}{=} \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_k}{\partial \bar{z}_1}(\zeta, z') \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z_1} \stackrel{(3)}{=} g_k. \end{aligned} \quad (13)$$

Differentiation under the integral sign is permitted as in the proof of *Proposition 1.3.23*. For z outside of the support of ρ , $\frac{\partial u}{\partial \bar{z}_k}(z) = 0$ for each $1 \leq k \leq n$, and so $u(z)$ is holomorphic for all $z \notin \text{supp}(\rho)$. Since ρ is compactly supported, for any $z' \in \mathbb{C}^{n-1}$, with $\|z'\|$ sufficiently large, $g_1(\zeta, z') = 0$ for any $\zeta \in \mathbb{C}$. So u is identically zero on the unbounded connected component Z of $G \setminus K$.

Define a function $\tilde{f} := (1 - \rho) \cdot f + u$ on G which, upon restricting to an open set $V \subset Z \cap (G \setminus \text{supp}(\rho)) \subset G \setminus K$, coincides with f . To see that \tilde{f} is holomorphic on G , we observe that

$$\frac{\partial \tilde{f}}{\partial \bar{z}_k} = (1 - \rho) \cdot \frac{\partial f}{\partial \bar{z}_k} - f \cdot \frac{\partial \rho}{\partial \bar{z}_k} + \frac{\partial u}{\partial \bar{z}_k} \stackrel{(13)}{=} (1 - \rho) \cdot \frac{\partial f}{\partial \bar{z}_k} - g_k + g_k = 0,$$

for all z in a neighbourhood of $G \setminus K$, where the last equality follows from the holomorphy of f on $G \setminus K$. Similarly, note that on U , $\rho \equiv 1$, so the above formal computation reduces to

$$\frac{\partial \tilde{f}}{\partial \bar{z}_k} = \frac{\partial u}{\partial \bar{z}_k} = g_k = f \cdot \frac{\partial \rho}{\partial \bar{z}_k} = 0.$$

Therefore, \tilde{f} is a holomorphic function on G and is an analytic extension of f in the sense of *Definition 1.3.1*, as required. \square

Remark 1.4.4. Let us make explicit that *Theorem 1.4.3* does not hold in one dimension. To see this, simply take G to be the entire complex plane and $K = \{0\}$. The function $f(z) = 1/z$ is holomorphic on all of $G \setminus K$ but does not extend to a function holomorphic on \mathbb{C} .

Remark 1.4.5. The assumption that $G \setminus K$ is connected is a necessary condition. To see this, for $n > 1$ let $K := \{z \in \mathbb{C}^n : |z| = \frac{1}{2}\}$ and $G = B^{(n)}(0, 1)$. Define the function $f : G \setminus K \rightarrow \mathbb{C}$ by

$$f(z) = \begin{cases} 1, & \frac{1}{2} < |z| < 1, \\ 0, & |z| < \frac{1}{2}. \end{cases}$$

It is clear that f is analytic on $G \setminus K$, but cannot be extended to a function analytic on G .

Remark 1.4.6. *Theorem 1.4.3* allows us to strengthen classical results of complex analysis in the higher-dimensional setting. For example, an immediate corollary of Hartogs' extension theorem is the following strengthening of Liouville's theorem:

Corollary 1.4.7. Let $K \subset \mathbb{C}^n$ be a compact set such that $\mathbb{C}^n \setminus K$ is connected. Suppose that $f : \mathbb{C}^n \setminus K \rightarrow \mathbb{C}$ is analytic and bounded. Then f is constant.

CHAPTER 2

The General Theory of Complex Analytic Spaces

The purpose of this chapter is to introduce the main objects of this exposition – the Stein spaces. These objects form a particular class of complex analytic spaces which, in the smooth category, were introduced by Karl Stein in his memorable work [62]. We develop the theory behind these general complex analytic spaces with motivation coming from the familiar geometric notion of a manifold.

§2.1. STEIN MANIFOLDS

We begin by surveying some results for domains in holomorphy in \mathbb{C}^n . This will serve as motivation for the definition of a Stein manifold which we introduce in this section.

Definition 2.1.1. Suppose that f is a holomorphic function on a domain G . Let w be a point on the boundary of G . We say that f is *completely singular* at w if for every connected neighbourhood $U \subseteq \mathbb{C}^n$ of w , and every connected component $D \subset U \cap G$, there does not exist a holomorphic function $g \in \mathcal{O}(U)$ such that $g|_D = f|_D$.

Example 2.1.2. Let $f(z) := \log(z)$ be the branch of the logarithm on $G := \mathbb{C} \setminus \mathbb{R}_-$. This function is completely singular only at the origin $z = 0$, but not at any other point along \mathbb{R}_- .

Definition 2.1.3. Let $G \subseteq \mathbb{C}^n$ be a domain. We say that G is a

- (i) *weak domain of holomorphy* if for each point $w \in \partial G$ there exists a function $f \in \mathcal{O}(G)$ which is completely singular at w .
- (ii) *domain of holomorphy* if there exists a function which is completely singular at every point of the boundary ∂G .

Example 2.1.4. It is clear that every domain in \mathbb{C} is a weak domain of holomorphy. Indeed, for any point $p \in \partial G$, the function $z \mapsto (z - p)^{-1}$ is completely singular at p . Moreover, the unit disk $\Delta(0, 1)$ is a domain of holomorphy since the function $z \mapsto \sum_{k=0}^{\infty} z^{k!}$ is completely singular at all points $z \in \mathbb{C}$ with $|z| = 1$.

Remark 2.1.5. It turns out that the notions of a weak domain of holomorphy and domain of holomorphy, as given in *Definition 2.1.3*, coincide. The proof of the fact that every weak domain of holomorphy is a domain of holomorphy is non-trivial. In the literature, this fits in the so-called *Levi problem* on the classification of domains of holomorphy based on properties of their boundaries. This phenomenon was first observed by E. E. Levi in [42] in some very special cases. For domains in \mathbb{C}^2 this was solved by K. Oka [51], and in $\mathbb{C}^{n>2}$ by Oka [53] (see also, e.g., [4], [49]).

A very useful characterisation of domains of holomorphy was given by H. Cartan and P. Thullen in [5]. Their characterisation is based on the notion of *holomorphic convexity*. Let us give the definition:

Definition 2.1.6. Let $G \subseteq \mathbb{C}^n$ be a domain, and $K \subset G$ a compact set. The *holomorphically convex hull* \widehat{K} of K is

$$\widehat{K} := \bigcap_{f \in \mathcal{O}(G)} \left\{ z \in G : |f(z)| \leq \max_{z \in K} |f(z)| \right\}.$$

We say that G is *holomorphically convex* if for every compact set $K \subset G$, the holomorphically convex hull $\widehat{K} \subset G$ is also compact.

Remark 2.1.7. To convince the reader that *holomorphic convexity* is an appropriate name, we note that the following characterisation of ordinary geometric convexity. Indeed, we say that a set $S \subseteq \mathbb{R}^n$ is *convex* if for all $x, y \in S$, and all $0 \leq t \leq 1$, the line segment $tx + (1-t)y \in S$. One may reformulate this however, as requiring that the convex hull $\text{conv}(K) \subset S$ is compact for all compact sets $K \subset S$. Further, recall that a function $T : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *affine* if there exists a linear function $A : \mathbb{R}^n \rightarrow \mathbb{R}$ and a vector $v \in \mathbb{R}^n$ such that $T(x) := A(x) + v$. Let $\text{Aff}(\mathbb{R}^n)$ denote the set of all affine functions on \mathbb{R}^n . One may show the following characterisation of the convex hull:

$$\text{conv}(K) = \bigcap_{T \in \text{Aff}(\mathbb{R}^n)} \left\{ x \in \mathbb{R}^n : |T(x)| \leq \max_{x \in K} |T(x)| \right\},$$

(see, e.g., [18, Chapter II.5]).

Lemma 2.1.8. Consider the sets $K \subset \mathbb{C}^n$ and $L \subset \mathbb{C}^m$. Then

- (i) $\widehat{K} \subset \mathbb{C}^n$ is closed. In particular, given any point $p \in \mathbb{C}^n \setminus \widehat{K}$, there exists an analytic function $f \in \mathcal{O}(\mathbb{C}^n)$ such that

$$\|f\|_K < 1 < |f(p)|.$$

- (ii) $\widehat{K \times L} \subset \widehat{K} \times \widehat{L}$.

PROOF. The fact that \widehat{K} is closed is clear. Moreover, if $p \notin \widehat{K}$, then for every $f \in \mathcal{O}(G)$ we have $|f(p)| > \|f\|_K$. So there exists a constant $\lambda \in \mathbb{R}$ such that $\|f\|_K < \lambda < |f(p)|$, and by normalising λ to 1 we obtain (i).

We observe that $\widehat{K \times \mathbb{C}^m} \subset \widehat{K} \times \mathbb{C}^m$ and $\widehat{\mathbb{C}^n \times L} \subset \mathbb{C}^n \times \widehat{L}$. Further, we have $K \times L \subset (K \times \mathbb{C}^m) \cap (\mathbb{C}^n \times L)$, and so

$$\widehat{K \times L} \subset (\widehat{K \times \mathbb{C}^m}) \cap (\widehat{\mathbb{C}^n \times L}) \subset (\widehat{K} \times \mathbb{C}^m) \cap (\mathbb{C}^n \times \widehat{L}) = \widehat{K} \times \widehat{L},$$

as required. □

We now state the theorem of Cartan and Thullen on domains of holomorphy, (see, e.g., [5], [18, p. 76–82], [59, p. 182–184]).

Theorem 2.1.9. (*Cartan–Thullen*). A domain $G \subseteq \mathbb{C}^n$ is a domain of holomorphy if and only if G is holomorphically convex.

Before introducing the definition of a Stein manifold, we remind the reader of the definition of a complex manifold.

Definition 2.1.10. Let X be a smooth manifold of dimension $2n$, and let $(U_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ a smooth atlas for X , i.e., X is covered by the open sets $(U_\lambda)_{\lambda \in \Lambda}$, and the *transition maps*

$$\varphi_\lambda \circ \varphi_\mu^{-1} : \varphi_\mu(U_\mu \cap U_\lambda) \longrightarrow \varphi_\lambda(U_\mu \cap U_\lambda)$$

are smooth in the sense of maps between open sets in \mathbb{C}^n . If these transition maps are holomorphic then we say that X is a *complex manifold* of *complex dimension* n . The pair $(U_\lambda, \varphi_\lambda)$ is called a *holomorphic chart* and an atlas consisting of holomorphic charts is referred to as a *holomorphic atlas*. Two holomorphic atlases on X are said to be *equivalent* if their union is again a holomorphic atlas for X .

Definition 2.1.11. A *complex manifold of dimension* n is a smooth manifold of (real) dimension $2n$ equipped with an equivalence class of holomorphic atlases.

Definition 2.1.12. Let X and Y be two complex manifolds of dimension m and n respectively. A map $f : X \longrightarrow Y$ is said to be *holomorphic* (or *analytic*) if for every chart $(U_\lambda, \varphi_\lambda)$ of X and every chart $(V_\lambda, \psi_\lambda)$ of Y such that $f(U_\lambda) \subset V_\lambda$, the map $\psi_\lambda \circ f \circ \varphi_\lambda^{-1}$ is a holomorphic map between open subsets of \mathbb{C}^m and \mathbb{C}^n . If $f : X \longrightarrow Y$ is holomorphic, bijective, and whose inverse $f^{-1} : Y \longrightarrow X$ is holomorphic, then f is said to be *biholomorphic*.

Remark 2.1.13. It is a well-known result that a bijective holomorphic function $f : X \longrightarrow Y$ automatically has a holomorphic inverse, (see, e.g., [22, p. 19]).

Note that in [62] the following class of manifolds are called *holomorphically complete*.

Definition 2.1.14. (*Stein*). Let X be an n -dimensional complex manifold. We say that X is a *Stein manifold* if

- (i) X is holomorphically convex.
- (ii) X is holomorphically separable.

For complex manifolds, the definition of holomorphic convexity is the straightforward generalisation of *Definition 2.1.6*. Indeed, a complex manifold X is said to be *holomorphically convex* if for every compact set $K \subset X$ the holomorphically convex hull

$$\widehat{K} = \bigcap_{f \in \mathcal{O}_X(X)} \left\{ z \in X : |f(z)| \leq \sup_{z \in K} |f(z)| \right\}$$

is compact. We say that X is *holomorphically separable* if for any two distinct points $x, y \in X$, there exists a holomorphic function $f \in \mathcal{O}_X(X)$ such that $f(x) \neq f(y)$.

Remark 2.1.15. The assumption of holomorphic separability expresses, in a vague sense, the ‘wealth of analytic functions’ defined on the complex manifold. The assumption of holomorphic separability informs us that no Stein manifold can be compact.

Remark 2.1.16. The motivation for such a class of complex manifolds (as described in, e.g., [30], [56]), originated from Stein’s interest in the Cousin problems, which we discuss in §4.3, (see also [11]). Indeed, throughout the former half of the twentieth century, it was understood that many problems in several complex variables, such as the first Cousin problems, could be solved on a class of complex manifolds which had similar properties to non-compact Riemann surfaces and domains of holomorphy.

Remark 2.1.17. Note that if M is a domain in \mathbb{C}^n , condition (ii) of *Definition 2.1.14* are trivially satisfied. Therefore, by the Cartan–Thullen theorem, the Stein manifolds in \mathbb{C}^n are exactly the domains of holomorphy. One may also show that any non-compact Riemann surface is a Stein manifold (see, e.g., [17, p. 205]).

§2.2. PRESHEAVES AND SHEAVES

Definition 2.2.1. A *presheaf of rings* \mathcal{R} on a topological space X consists of the following data.

- (i) To every open set U , there is an assigned ring $\mathcal{R}(U)$.
- (ii) Given an inclusion of open sets $U \subset V \subset X$, there is an induced ring homomorphism $\text{res}_U^V : \mathcal{R}(V) \rightarrow \mathcal{R}(U)$ called a *restriction map*.
- (iii) This data satisfies:
 - (a) If $U \subset V \subset W \subset X$ is an inclusion of open sets, then $\text{res}_U^W = \text{res}_U^V \circ \text{res}_V^W$.
 - (b) $\text{res}_U^U = \text{Id}_U$.
 - (c) $\mathcal{R}(\emptyset) = \{0\}$.

Note that by replacing the term “ring” in *Definition 2.2.1* with “set”, “abelian group”, etc., we obtain a presheaf of sets, a presheaf of abelian groups, etc. In the circumstance that a particular definition or result does not depend on what type of presheaf we consider, we will simply write that \mathcal{R} is a presheaf on X .

Let \mathcal{R} be a presheaf on X , and $U \subset X$ an open subset of a topological space X . An element of $\mathcal{R}(U)$ is called a *section* of \mathcal{R} over U . Moreover, for $x \in X$, we set

$$\mathcal{R}_x := \varinjlim_U \mathcal{R}(U),$$

where the above right-hand-side is the direct limit over all open neighbourhoods U of x . The ring \mathcal{R}_x is called the *stalk* of \mathcal{R} at x , and the elements of \mathcal{R}_x are called *germs*.

Definition 2.2.2. Fix a topological space X and let \mathcal{R} and \mathcal{S} be two presheaves on X . For convenience, assume that \mathcal{R} and \mathcal{S} are presheaves of abelian groups¹. A *morphism of presheaves* $f : \mathcal{R} \rightarrow \mathcal{S}$ is a collection of group homomorphisms $f_U : \mathcal{R}(U) \rightarrow \mathcal{S}(U)$, where $U \subseteq X$ is open, such that for any inclusion $V \subseteq U$, the diagram

¹This definition may be equally applied to sheaves of sets, rings, etc.

$$\begin{array}{ccc}
\mathcal{R}(U) & \xrightarrow{f_U} & \mathcal{S}(U) \\
\text{res}_V^U \downarrow & & \downarrow \text{res}_V^U \\
\mathcal{R}(V) & \xrightarrow{f_V} & \mathcal{S}(V)
\end{array}$$

commutes.

Definition 2.2.3. Let $f : \mathcal{R} \rightarrow \mathcal{S}$ be a morphism of presheaves over a topological space X . For each open set $U \subset X$, let $f_U : \mathcal{R}(U) \rightarrow \mathcal{S}(U)$ be the associated map on sections. We define the following presheaves:

- (i) the *presheaf kernel* $\mathcal{Ker}(f)$, $U \mapsto \ker(f_U)$.
- (ii) the *presheaf cokernel* $\mathcal{Coker}(f)$, $U \mapsto \text{coker}(f_U) := \mathcal{S}(U)/\text{im}(f_U)$.

Note that a morphism of presheaves $f : \mathcal{R} \rightarrow \mathcal{S}$ is said to be *injective* (resp. *surjective*, resp. an *isomorphism*) if $f_U : \mathcal{R}(U) \rightarrow \mathcal{S}(U)$ is injective (resp. surjective, resp. an isomorphism) for every open set $U \subset X$.

Definition 2.2.4. A *sheaf* \mathcal{R} on a topological space X is a presheaf which satisfies the following two additional constraints. Let U be any open set in X , with $(U_\lambda)_{\lambda \in \Lambda}$ an open cover of U .

- (i) (Uniqueness). If $f \in \mathcal{R}(U)$ satisfies $\text{res}_{U_\lambda}^U(f) = 0$ for all $\lambda \in \Lambda$. Then $f = 0$ in $\mathcal{R}(U)$.
- (ii) (Existence). If we are given a collection $f_\lambda \in \mathcal{R}(U_\lambda)$ such that for all pairs $\mu, \lambda \in \Lambda$,

$$\text{res}_{U_\lambda \cap U_\mu}^{U_\lambda}(f_\lambda) = \text{res}_{U_\lambda \cap U_\mu}^{U_\mu}(f_\mu).$$

Then there exists some $f \in \mathcal{R}(U)$ such that $\text{res}_{U_\lambda}^U(f) = f_\lambda$ for all $\lambda \in \Lambda$.

Remark 2.2.5. Note that conditions (i) and (ii) are equivalent to saying that for any open set $U \subseteq X$, and any open cover $\mathcal{U} := (U_\lambda)_{\lambda \in \Lambda}$ of U , stable under finite intersections, the morphism

$$\mathcal{R}(U) \longrightarrow \varprojlim_{\lambda} \mathcal{R}(U_\lambda)$$

is an isomorphism.

Example 2.2.6. Let X be a complex manifold. The most important example of a sheaf, for our purposes at least, is the *sheaf* \mathcal{O}_G of *analytic functions* on a domain $G \subseteq X$. Other examples of presheaves and sheaves that will appear in this exposition include:

- the *sheaf* \mathcal{C}_X of *continuous functions* which assigns to each open subset $U \subseteq X$, the ring

$$\mathcal{C}_X(U) := \{f : U \rightarrow \mathbb{C} : f \text{ is a continuous function}\}.$$

- the *sheaf* \mathcal{C}_X^∞ of *smooth functions* which assigns to each open subset $U \subseteq X$, the ring

$$\mathcal{C}_X^\infty(U) := \{f : U \rightarrow \mathbb{C} : f \text{ is a smooth function}\}.$$

- the *constant presheaf* which is denoted by R which assigns to each open set $U \subseteq X$ simply the ring R . If R is the zero ring, we refer to the constant presheaf as the *zero presheaf*.

Notational Remark 2.2.7. Note that to simplify notation, in many instances we will \mathcal{O} in place of \mathcal{O}_G . Moreover, since the stalk of \mathcal{O} at $p \in G$ is the ring of convergent power series $\mathbb{C}\{z\}$ centred at p , we often use the notation \mathcal{O}_p and $\mathbb{C}\{z\}$ interchangeably without explicit mention.

Definition 2.2.8. Let \mathcal{R} and \mathcal{S} be two sheaves on a topological space X . A *morphism of sheaves* $f : \mathcal{R} \rightarrow \mathcal{S}$ is defined to be a morphism of underlying presheaves.

Similarly, a morphism of sheaves $f : \mathcal{R} \rightarrow \mathcal{S}$ is said to be *injective* (resp. *surjective*, resp. an *isomorphism*) if the morphism of underlying presheaves is injective (resp. surjective, resp. an isomorphism).

Further, if $f : \mathcal{R} \rightarrow \mathcal{S}$ is a morphism of sheaves, we define the presheaf kernel and the presheaf cokernel by the same formulas as those exhibited in *Definition 2.2.3*.

Proposition 2.2.9. Let \mathcal{R} and \mathcal{S} be two sheaves of abelian groups over a topological space X . Let $f : \mathcal{R} \rightarrow \mathcal{S}$ be a morphism of sheaves. Then f is an isomorphism of sheaves if and only if the induced map on stalks $f_x : \mathcal{R}_x \rightarrow \mathcal{S}_x$ is an isomorphism of abelian groups for each $x \in X$.

PROOF. It is clear that if $f : \mathcal{R} \rightarrow \mathcal{S}$ is an isomorphism of sheaves, then $f_x : \mathcal{R}_x \rightarrow \mathcal{S}_x$ is an isomorphism for each $x \in X$. We therefore concern ourselves only with the converse statement. Suppose that $f_x : \mathcal{R}_x \rightarrow \mathcal{S}_x$ is an isomorphism for each $x \in X$. For each open set $U \subseteq X$, let $f_U : \mathcal{R}(U) \rightarrow \mathcal{S}(U)$ denote the associated map on sections. It suffices to show that f_U is an isomorphism for each open set $U \subseteq X$. To this end, we will first show that f_U is injective. Suppose that the section $s \in \mathcal{R}(U)$ lies in the kernel of f_U . Denote by s_p the stalk of s at $p \in X$. Then for each $x \in U$,

$$(f_U(s))_x = f_x(s_x) = 0 \quad \text{in } \mathcal{S}_x.$$

Since f_x is injective, $s_x = 0$ in \mathcal{R}_x for each $x \in U$. In other words, there is an open neighbourhood $V_x \subset U$ of x such that $s|_{V_x} = 0$. Since we may cover U by the neighbourhoods $(V_x)_{x \in U}$, (i) of *Definition 2.2.4* implies that $s = 0$ in U . This proves that f_U is injective.

To see that f_U is surjective, let $t \in \mathcal{S}(U)$, and for each $x \in U$, denote by t_x the germ of t at x in \mathcal{S}_x . Let $s_x \in \mathcal{R}_x$ denote the stalk of a section $s \in \mathcal{R}(V_x)$, where $V_x \subset U$ is a neighbourhood of x , such that $f_x(s_x) = t_x$. In other words, the germs of $f_U(s)$ and t , at x , coincide. We choose V_x sufficiently small such that $f_U(s)|_{V_x}$ and $t|_{V_x}$ represent the same element in $\mathcal{S}(V_x)$. As we did before, we cover U by the open neighbourhoods V_x . On each of these we have $s(x) \in \mathcal{R}(V_x)$. Now for two points $z, w \in X$, $s(z)|_{V_z \cap V_w}$ and $s(w)|_{V_z \cap V_w}$ are both mapped to $t|_{V_z \cap V_w}$ by $f|_{V_z \cap V_w}$. We have shown that for every open set $U \subseteq X$, f_U is injective, so

$$s(z)|_{V_z \cap V_w} = s(w)|_{V_z \cap V_w}.$$

Then by (ii) of *Definition 2.2.4*, we obtain a section $s \in \mathcal{R}(U)$ such that $s|_{V_z} = s(z)$. In particular, this shows that f_U is surjective and completes the proof. \square

Proposition 2.2.10. Let \mathcal{S} be a presheaf on a topological space X . We may associate to \mathcal{S} a sheaf $\overline{\mathcal{S}}$ such that, for any sheaf \mathcal{R} on X , and any morphism of presheaves $f : \mathcal{S} \rightarrow \mathcal{R}$, there exists a unique morphism of sheaves $h : \overline{\mathcal{S}} \rightarrow \mathcal{R}$ such that $f = h \circ g$, where $g : \mathcal{S} \rightarrow \overline{\mathcal{S}}$.

PROOF. The construction of $\overline{\mathcal{S}}$ is obvious. Indeed, given any open set $U \subset X$, we set $\overline{\mathcal{S}}(U)$ to be the set of all functions $f : U \rightarrow \bigcup_{x \in U} \mathcal{S}_x$ such that

- (i) $s(x) \in \mathcal{S}_x$ for all $x \in U$, and

- (ii) for every point $x \in U$, there is an open neighbourhood $V \subset U$ of x , and an element $t \in \mathcal{S}(V)$ such that the germ of t at y coincides with $s(y)$ for all $y \in V$.

The properties of $\overline{\mathcal{S}}$, and the uniqueness of h , in the statement of the proposition are now immediate. \square

To break away from all this abstract nonsense, we consider the following concrete example.

Example 2.2.11. Let \mathcal{S} denote the presheaf on \mathbb{R} which, on any open set $U \subseteq \mathbb{R}$ not containing both 0 and 1, coincides with the sheaf $\mathcal{C}_{\mathbb{R}}$ of continuous functions. For an open set U containing 0 and 1 however, \mathcal{S} assigns to U the set of continuous functions $f : U \rightarrow \mathbb{R}$ such that $f(0) = f(1)$. For an open set $U \subseteq \mathbb{R}$ such that $0 \in U$, but $1 \notin U$, then $\mathcal{S}|_U$ coincides with the sheaf $\mathcal{C}_{\mathbb{R}}$ of continuous functions on \mathbb{R} . The same is of course true if we take $U \subset \mathbb{R}$ such that $1 \in U$, but $0 \notin U$. Hence at least locally, this presheaf \mathcal{S} is simply $\mathcal{C}_{\mathbb{R}}$. Therefore, as mentioned above, to form the sheaf $\overline{\mathcal{S}}$ we include enough sections which will allow us to glue local sections and remove the non-zero global sections which are locally zero. It is clear that the resulting sheaf which we obtain is $\overline{\mathcal{S}} = \mathcal{C}_{\mathbb{R}}$.

Proposition 2.2.12. Let $f : \mathcal{R} \rightarrow \mathcal{S}$ be a morphism of sheaves over X . The presheaf kernel $\mathcal{K}er(f)$ is a sheaf.

PROOF. We need to show that $\mathcal{K}er(f)$ satisfies conditions (i) and (ii) of *Definition 2.2.4*. To this end, let $U \subseteq X$ be an open set, covered by the open sets $(U_\lambda)_{\lambda \in \Lambda}$. We first show condition (i) of *Definition 2.2.4*. Suppose that $s \in \mathcal{K}er(f)(U)$ is such that $s|_{U_\lambda} = 0$ for all $\lambda \in \Lambda$. Since $\mathcal{K}er(f)(U) \subseteq \mathcal{R}(U)$, we may identify s as an element of $\mathcal{R}(U)$ which satisfies $s|_{U_\lambda} = 0$ for all $\lambda \in \Lambda$. Since \mathcal{R} is a sheaf, \mathcal{R} satisfies (i) of *Definition 2.2.4*, and therefore $s = 0$ in $\mathcal{R}(U)$. Then $f_U(s) = f_U(0) = 0$ shows that $s \in \ker(f)(U)$ and since $s = 0$ in $\mathcal{R}(U)$, we conclude that $s = 0$ in $\mathcal{K}er(f_U)$.

To establish condition (ii) of *Definition 2.2.4*, suppose that we have local sections $s_\lambda \in \mathcal{K}er(f)(U_\lambda)$ such that

$$s_\lambda|_{U_\lambda \cap U_\mu} = s_\mu|_{U_\lambda \cap U_\mu}, \quad \forall \lambda, \mu \in \Lambda.$$

Then these local sections, viewed as sections of $\mathcal{R}(U)$, can be glued together to form a section $s \in \mathcal{R}(U)$ which satisfies $s|_{U_\lambda} = s_\lambda$. Since $\mathcal{K}er(f)(U) \subseteq \mathcal{R}(U)$, we identify s with an element of the subgroup $\mathcal{K}er(f)(U)$, and this completes the proof. \square

We note that it is not true in general that $\mathcal{C}oker(f)$ is a sheaf. Note however, that for all $x \in X$,

$$(\mathcal{K}er(f))_x = \ker(f_x), \quad \text{and} \quad (\mathcal{C}oker(f))_x = \mathcal{C}oker(f_x).$$

In particular, we may equivalently define a morphism of sheaves $f : \mathcal{R} \rightarrow \mathcal{S}$ to be injective (resp. surjective, resp. an isomorphism) if and only if $f_x : \mathcal{R}_x \rightarrow \mathcal{S}_x$ is injective (resp. surjective, resp. an isomorphism) for each $x \in X$.

Definition 2.2.13. Let $(\mathcal{S}^q)_{q \in \mathbb{N}}$ denote a collection of sheaves on a topological space X . A sequence of morphisms of sheaves

$$\mathcal{S}^1 \xrightarrow{\alpha^1} \mathcal{S}^2 \xrightarrow{\alpha^2} \mathcal{S}^3 \rightarrow \dots$$

is said to be *exact* if and only $\ker(\alpha_x^{q+1}) = \text{im}(\alpha_x^q)$ for each $q \geq 1$, and all $x \in X$. Here, α_x^q denotes the associated morphism at the level of stalks, i.e., $\alpha_x^q : \mathcal{S}_x^q \rightarrow \mathcal{S}_x^{q+1}$.

Many of the notions familiar to us for abelian groups can be generalised to sheaves of abelian groups. We mention some of these constructions in the following definition; details of the constructions are omitted. For details, see, e.g., [28, Chapter 2.1], [26, Chapter IV.A], [37, Chapter 3], [19, Chapter A].

Definition 2.2.14. Let \mathcal{R} and \mathcal{S} be two sheaves of abelian groups on a fixed topological space X .

- (i) We say that \mathcal{S} is a *subsheaf* of \mathcal{R} , and write $\mathcal{S} \subseteq \mathcal{R}$, if \mathcal{S}_x is a subgroup of \mathcal{R}_x for each $x \in X$.
- (ii) We say that \mathcal{I} is an *ideal sheaf* if \mathcal{I}_x is an ideal in \mathcal{R}_x for each $x \in X$.
- (iii) The restriction of \mathcal{S} to an open set $U \subseteq X$, denoted by \mathcal{S}_U , is the sheaf associated to the presheaf $V \mapsto \mathcal{S}(V)$, where V is an open subset of U .
- (iv) The direct sum of \mathcal{R} and \mathcal{S} , written $\mathcal{R} \oplus \mathcal{S}$, is the sheaf associated to the presheaf $U \mapsto \mathcal{R}(U) \oplus \mathcal{S}(U)$.
- (v) If \mathcal{S} is a subsheaf of \mathcal{R} , the *quotient sheaf* \mathcal{R}/\mathcal{S} is the sheaf associated to the presheaf $U \mapsto \mathcal{R}(U)/\mathcal{S}(U)$.

Remark 2.2.15. We offer some further remarks on the definition of the quotient sheaf; we maintain the above notation. The formula $U \mapsto \mathcal{R}(U)/\mathcal{S}(U)$ will not in general define a sheaf, only a presheaf. Therefore, to obtain a sheaf, we must sheafify. It will be useful to elaborate a little on the sheafification process. Indeed, if we cover X by the open sets $(U_\lambda)_{\lambda \in \Lambda}$, a local section of the quotient presheaf is an element $f_\lambda \in \mathcal{R}(U_\lambda)/\mathcal{S}(U_\lambda)$. To form a sheaf, we need to be able to glue these local sections and obtain a global section. More precisely, we require that for any two local sections $f_\lambda \in \mathcal{R}(U_\lambda)/\mathcal{S}(U_\lambda)$, and $f_\mu \in \mathcal{R}(U_\mu)/\mathcal{S}(U_\mu)$,

$$f_\lambda|_{U_\lambda \cap U_\mu} - f_\mu|_{U_\lambda \cap U_\mu} = 0$$

in $\mathcal{R}(U_\lambda \cap U_\mu)/\mathcal{S}(U_\lambda \cap U_\mu)$. This may be equivalently expressed as $f_\lambda|_{U_\lambda \cap U_\mu} - f_\mu|_{U_\lambda \cap U_\mu} \in \mathcal{S}(U_\lambda \cap U_\mu)$.

Definition 2.2.16. A *ringed space* is a pair (X, \mathcal{R}) consisting of a topological space X and a sheaf of rings \mathcal{R} over X . If each stalk of \mathcal{R} is a local ring, we call (X, \mathcal{R}) a *locally ringed space*. If each stalk \mathcal{R}_x is an analytic \mathbb{C} -algebra, then (X, \mathcal{R}) is said to be a *\mathbb{C} -analytic ringed space*. The sheaf \mathcal{R} associated to the ringed space (X, \mathcal{R}) is referred to as the *structure sheaf* of X . We will often write X in place of (X, \mathcal{R}) .

Definition 2.2.17. Let (X, \mathcal{R}) and (Y, \mathcal{S}) be two ringed spaces. A *morphism of ringed spaces* is a pair $(f, f^\flat) : (X, \mathcal{R}) \rightarrow (Y, \mathcal{S})$ such that

- (i) $f : X \rightarrow Y$ is continuous.
- (ii) for every open set $U \subset V$, there is an induced ring homomorphism $f_U^\flat : \mathcal{S}(U) \rightarrow \mathcal{R}(f^{-1}(U))$.
- (iii) the maps f^\flat must be compatible with the restriction maps of \mathcal{R} and \mathcal{S} . That is, for any inclusion of open sets $V \subseteq U \subseteq Y$, the diagram

$$\begin{array}{ccc}
\mathcal{S}(U) & \xrightarrow{f_U^b} & \mathcal{R}(f^{-1}(U)) \\
\text{res}_V^U \downarrow & & \downarrow \text{res}_{f^{-1}(V)}^{f^{-1}(U)} \\
\mathcal{S}(V) & \xrightarrow{f_V^b} & \mathcal{R}(f^{-1}(V))
\end{array}$$

commutes.

Definition 2.2.18. Let (X, \mathcal{R}) and (Y, \mathcal{S}) be two locally ringed spaces. A *morphism of locally ringed spaces* is a morphism of ringed spaces such that, for each $p \in X$, the induced map on stalks $f_p^b : \mathcal{S}_{f(p)} \rightarrow \mathcal{R}_p$, is a morphism of local rings, i.e., the maximal ideal in \mathcal{R}_p contains the image of the maximal ideal in $\mathcal{S}_{f(p)}$ under f_p^b . A *morphism of \mathbb{C} -analytic ringed spaces* is a morphism of ringed spaces such that, for each $p \in X$, the induced map on stalks, $f_p^b : \mathcal{S}_{f(p)} \rightarrow \mathcal{R}_p$, is a morphism of analytic \mathbb{C} -algebras.

§2.3. COHERENCE THEORY OF SHEAVES – THE STATEMENTS

The sheaves discussed in the previous section allow us to pass from local phenomena to global phenomena. The coherent sheaves discussed in this section allow us to pass from punctual phenomena² to local phenomena. For example, suppose that $\mathcal{S}' \rightarrow \mathcal{S} \rightarrow \mathcal{S}''$ is a sequence of coherent sheaves (whatever this means) on a topological space X . If for some point $x \in X$, the sequence $\mathcal{S}'_x \rightarrow \mathcal{S}_x \rightarrow \mathcal{S}''_x$ is exact, then for every point in a sufficiently small neighbourhood of x , the sequence remains exact.

Definition 2.3.1. Let \mathcal{R} be a sheaf of rings and \mathcal{S} a sheaf of abelian groups on a topological space X . We say that \mathcal{S} is a *sheaf of \mathcal{R} -modules* (or an *\mathcal{R} -module*) if there is a morphism of sheaves $\mathcal{R} \oplus \mathcal{S} \rightarrow \mathcal{S}$ which descends to a morphism on stalks $\mathcal{R}_x \oplus \mathcal{S}_x \rightarrow \mathcal{S}_x$, endowing \mathcal{S}_x with the structure of an \mathcal{R}_x -module for each $x \in X$.

Definition 2.3.2. Let \mathcal{S} and \mathcal{T} be two sheaves of \mathcal{R} -modules over a fixed topological space X . A *morphism of sheaves of \mathcal{R} -modules* (or an *\mathcal{R} -morphism*) is a morphism of sheaves $f : \mathcal{S} \rightarrow \mathcal{T}$ such that $f_x : \mathcal{S}_x \rightarrow \mathcal{T}_x$ is a morphism of \mathcal{R}_x -modules for each $x \in X$. If the sheaf of rings is taken to be the sheaf \mathcal{O} of analytic functions on a domain in \mathbb{C}^n , or more generally, on a complex analytic space (whatever these are), then a sheaf \mathcal{S} of \mathcal{O} -modules is called an *analytic sheaf*.

Definition 2.3.3. Let \mathcal{R} be a sheaf of rings, and \mathcal{S}, \mathcal{T} two sheaves of \mathcal{R} -modules over a topological space X . We define

- (i) the *Hom sheaf*, denoted by $\mathcal{H}om_{\mathcal{R}}(\cdot, \cdot)$, to be the sheaf which assigns to each open set $U \subset X$ the ring $U \mapsto \text{Hom}(\mathcal{S}_U, \mathcal{T}_U)$.
- (ii) the *dual sheaf* of \mathcal{S} , denoted by \mathcal{S}^* , to be the sheaf $U \mapsto \mathcal{S}^*(U) := \mathcal{H}om_{\mathcal{R}_U}(\mathcal{S}_U, \mathcal{R}_U)$.
- (iii) the *tensor product of \mathcal{S} and \mathcal{T} over \mathcal{R}* to be the sheaf associated to the presheaf $U \mapsto \mathcal{S}(U) \otimes_{\mathcal{R}(U)} \mathcal{T}(U)$.

For the proof of the fact that $\mathcal{H}om_{\mathcal{R}}(\cdot, \cdot)$ defines a sheaf and not just a presheaf, see, e.g., [48, p. 176].

²i.e., phenomena that occur at the level of points.

Definition 2.3.4. Let \mathcal{R} be a sheaf of rings on a topological space X . Let \mathcal{S} and \mathcal{T} be two sheaves of \mathcal{R} -modules with $\mu : \mathcal{S} \rightarrow \mathcal{T}$ a morphism of \mathcal{R} -modules. A *syzygy* for μ is a morphism of \mathcal{R} -modules $\lambda : \mathcal{R}^p \rightarrow \mathcal{S}$ such that the sequence

$$\mathcal{R}^p \xrightarrow{\lambda} \mathcal{S} \xrightarrow{\mu} \mathcal{T}$$

is an exact sequence. A *chain of syzygies* for an \mathcal{R} -module \mathcal{S} is an exact sequence of \mathcal{R} -modules

$$\dots \rightarrow \mathcal{R}^{p_1} \xrightarrow{\lambda_1} \mathcal{R}^p \xrightarrow{\lambda} \mathcal{S} \rightarrow 0,$$

where each λ_j is a morphism of \mathcal{R} -modules.

Definition 2.3.5. We say that a chain of syzygies for an analytic sheaf \mathcal{R} *terminates at the k th step* if the sequence

$$0 \rightarrow \mathcal{O}^{p_k} \xrightarrow{\lambda_k} \mathcal{O}^{p_{k-1}} \rightarrow \dots \rightarrow \mathcal{O}^{p_1} \xrightarrow{\lambda_1} \mathcal{O}^p \xrightarrow{\lambda} \mathcal{R} \rightarrow 0 \quad (14)$$

is an exact sequence of analytic sheaves, and each λ_j is a morphism of analytic sheaves.

The following theorem tells us that if \mathcal{S} is an analytic sheaf on a domain $G \subseteq \mathbb{C}^n$ then for any point $p \in G$, there is an open neighbourhood of p , over which \mathcal{S} admits a terminating chain of syzygies.

Theorem 2.3.6. (*Hilbert's syzygy theorem for analytic sheaves*). Let U be an open neighbourhood of the origin in \mathbb{C}^n , and let $\mu_1 : \mathcal{O}_U^{p_1} \rightarrow \mathcal{O}_U^q$ be a morphism of analytic sheaves. Then there exists an open neighbourhood $V \subset U$ of the origin, over which, we have a terminating chain of syzygies of the form

$$0 \rightarrow \mathcal{O}_V^{p_n} \xrightarrow{\mu_n} \dots \xrightarrow{\mu_2} \mathcal{O}_V^{p_1} \xrightarrow{\mu_1} \mathcal{O}_V^q.$$

Definition 2.3.7. Fix a topological space X and let \mathcal{R} be a sheaf of rings on X . A sheaf of \mathcal{R} -modules \mathcal{S} is said to be of *finite type* if it is locally the image of a finitely generated free \mathcal{R} -module. That is, for each $x \in X$, there is an open neighbourhood $U \subseteq X$ of x and an exact sequence of the form $\mathcal{R}_U^p \rightarrow \mathcal{S}_U \rightarrow 0$, for some $p \in \mathbb{N}$.

Remark 2.3.8. Note that this may be equivalently expressed as every stalk \mathcal{S}_x being a finitely generated \mathcal{R}_x -module with the generators depending continuously on x .

Definition 2.3.9. A sheaf of \mathcal{R} -modules \mathcal{S} of finite type is said to be a *coherent \mathcal{R} -module* if for any open set $U \subseteq X$, and any finite set of sections $s_1, \dots, s_p \in \mathcal{S}(U)$, the *sheaf of relations* $\mathcal{R}\text{el}(s_1, \dots, s_p)$ over U is of finite type. Moreover, by \mathcal{R} being a *coherent sheaf of rings* we mean that \mathcal{R} is a coherent sheaf of \mathcal{R} -modules.

In more detail, fix a neighbourhood U of a point $x \in X$. Given finitely many sections $s_1, \dots, s_p \in \mathcal{S}(U)$, we define a morphism of sheaves $\sigma_U : \mathcal{R}_U^p \rightarrow \mathcal{S}_U$,

$$\sigma_U((\lambda_1)_x, \dots, (\lambda_p)_x) := \sum_{k=1}^p (\lambda_k)_x (s_k)_x, \quad x \in U.$$

If \mathcal{S} is a sheaf of finite type, for each point $x \in X$, there exists a neighbourhood U of x such that σ_U is surjective. The kernel of this morphism of sheaves is a sheaf by *Proposition 2.2.12*, and we refer to this sheaf

$\mathcal{R}\text{el}(s_1, \dots, s_p)$ as the *sheaf of relations*.

We offer a brief survey of some useful results on coherent sheaves.

Theorem 2.3.10. ([37, p. 140], [19, p. 11]). Let \mathcal{R} be a sheaf of rings on a topological space X , and let

$$0 \longrightarrow \mathcal{S}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}'' \longrightarrow 0$$

be an exact sequence of \mathcal{R} -modules. If any two of the sheaves in the above sequence are coherent, the third is also coherent.

Proposition 2.3.11. ([37, p. 141], [19, p. 12]).

- (i) Let $\varphi : \mathcal{S} \rightarrow \mathcal{S}'$ be an \mathcal{R} -morphism of coherent sheaves \mathcal{S} and \mathcal{S}' . Then $\mathcal{H}\text{er}(\varphi)$, $\mathcal{I}\text{m}(\varphi)$, and $\mathcal{C}\text{oker}(\varphi)$ are coherent sheaves of \mathcal{R} -modules.
- (ii) A subsheaf \mathcal{J} of a coherent sheaf is coherent if and only if \mathcal{J} is a sheaf of finite type.
- (iii) The direct sum of finitely many coherent sheaves is coherent.
- (iv) If \mathcal{I}_1 and \mathcal{I}_2 are two coherent ideal \mathcal{R} -sheaves, then $\mathcal{I}_1 \cdot \mathcal{I}_2$ is also coherent.
- (v) Let \mathcal{R} be a coherent sheaf of rings. Then the sheaf of \mathcal{R} -modules \mathcal{S} is coherent if and only if, for every $x \in X$, there exists a neighbourhood U of x , with positive integers p, q , and an exact sequence

$$\mathcal{R}_U^q \longrightarrow \mathcal{R}_U^p \longrightarrow \mathcal{S}_U \longrightarrow 0.$$

The following definition will be important to our discussion of complex analytic spaces.

Definition 2.3.12. Let \mathcal{R} be a sheaf of rings on a topological space X . We let $\mathcal{N} := \mathcal{N}(\mathcal{R})$ denote the *nilradical sheaf (of \mathcal{R})*. That is, \mathcal{N} is the sheaf whose stalk \mathcal{N}_p is the nilradical of \mathcal{R}_p , i.e.,

$$\mathcal{N}(\mathcal{R}) := \bigcup_{p \in X} \mathcal{N}(\mathcal{R}_p) \subset \mathcal{R}.$$

The quotient sheaf $\text{Red}(\mathcal{R}) := \mathcal{R}/\mathcal{N}$ is called the *reduction of \mathcal{R}* .

Theorem 2.3.13. ([19, p. 12]). Let \mathcal{R} and \mathcal{J} be coherent sheaf of \mathcal{R} -modules, and \mathcal{S} a sheaf of \mathcal{R}/\mathcal{J} -modules. Then \mathcal{S} is coherent as a sheaf of \mathcal{R}/\mathcal{J} -modules if and only if it is coherent as a sheaf of \mathcal{R} -modules. In particular,

- (i) \mathcal{R}/\mathcal{J} is a coherent sheaf of rings.
- (ii) if \mathcal{R} and the nilradical \mathcal{N} of \mathcal{R} are coherent, then $\text{red } \mathcal{R} = \mathcal{R}/\mathcal{N}$ is a coherent sheaf.

§2.4. COMPLEX ANALYTIC SPACES

We now have the necessary machinery required to introduce the formidable complex analytic spaces. In a crude sense, they are complex manifolds which admit singularities. The construction is analogous to Grothendieck's notion of a scheme in algebraic geometry [23]. We offer three equivalent definitions of (reduced) complex analytic spaces, following the treatment given in [24]. We begin with a discussion of the local picture of these complex analytic spaces.

We assume throughout this section unless otherwise stated, that all topological spaces are Hausdorff and second countable.

Definition 2.4.1. Let $G \subseteq \mathbb{C}^n$ be a domain. An *analytic set* (or *analytic variety*) in G is a set A which is locally given by the common zero set of a finite number of analytic functions. That is, for any point $x \in G$, there exists an open neighbourhood $U \subset G$ and a finite number of analytic functions $f_1, \dots, f_k \in \mathcal{O}(U)$ such that

$$A \cap U = V(f_1, \dots, f_k) \cap U = \{z \in U : f_j(z) = 0, \forall 1 \leq j \leq k\}.$$

Remark 2.4.2. Note that any domain $G \subseteq \mathbb{C}^n$ can be realised as the analytic set given by the vanishing of the function $f : G \rightarrow \mathbb{C}$ which is identically zero.

Definition 2.4.3. Let A and B be analytic sets in the respective domains $G \subseteq \mathbb{C}^n$ and $D \subseteq \mathbb{C}^m$. We say that $f : A \rightarrow B$ is a *morphism* (or *analytic sets*) if f is the restriction to A of a holomorphic map $\tilde{f} : U \rightarrow V$, where U and V are open neighbourhoods of A and B respectively.

We denote by (A, \mathcal{O}_A) the \mathbb{C} -analytic ringed space whose underlying topological space is A and whose structure sheaf is the sheaf of morphisms $f : A \rightarrow \mathbb{C}$.

Definition 2.4.4. Let X be a topological space. An *analytic atlas* (or *holomorphic atlas*) is a collection $(U_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$, where $(U_\lambda)_{\lambda \in \Lambda}$ is an open cover of X , and for each $\lambda \in \Lambda$ the map $\varphi_\lambda : U_\lambda \rightarrow A_\lambda$ is a homeomorphism onto a locally closed analytic set $A_\lambda \subset \mathbb{C}^{n_\lambda}$. Moreover, for each pair $\mu, \lambda \in \Lambda$, the transition maps

$$\varphi_\mu \circ \varphi_\lambda^{-1} : \varphi_\lambda(U_\lambda \cap U_\mu) \rightarrow \varphi_\mu(U_\lambda \cap U_\mu)$$

are morphisms of analytic sets. Each $(U_\lambda, \varphi_\lambda)$ is called an *analytic chart*. Two analytic atlases on X are said to be *equivalent* if their union is again an analytic atlas for X .

Definition 2.4.5. A (*reduced*) *complex analytic space* is a (Hausdorff, second countable) topological space X together with an equivalence class of analytic atlases.

Definition 2.4.6. Let X and Y be two reduced complex analytic spaces. A map $f : X \rightarrow Y$ is called a *morphism of reduced complex analytic spaces* (or *holomorphic map*) if for every analytic chart $(U_\lambda, \varphi_\lambda)$ of X , and every chart (V_μ, ψ_μ) of Y , such that $f(U_\lambda) \subset V_\mu$, the map $\psi_\mu \circ f \circ \varphi_\lambda^{-1}$ is a morphism of analytic sets. If f is bijective with inverse $f^{-1} : Y \rightarrow X$ also a morphism of reduced complex analytic spaces, we say that f is an *isomorphism* (of reduced complex analytic spaces) (or *biholomorphic map*).

Remark 2.4.7. Let us mention explicitly that by an *analytic function* on a reduced complex analytic space, we mean a morphism $f : X \rightarrow \mathbb{C}$, with \mathbb{C} viewed as a complex analytic space.

We offer the following useful alternative definition.

Definition 2.4.8. A (reduced) complex analytic space is a \mathbb{C} -analytic ringed space (X, \mathcal{O}_X) , where X is a (Hausdorff, second countable) topological space and \mathcal{O}_X is a subsheaf of \mathcal{C}_X which satisfies the following condition: For each point $x \in X$, there is an open neighbourhood $U \subseteq X$ of x such that $(U, \mathcal{O}_X|_U)$ is isomorphic (as a \mathbb{C} -analytic ringed space) to (A, \mathcal{O}_A) , where A is a locally closed analytic set in some \mathbb{C}^n . We call $(U, \mathcal{O}_X|_U)$ a local model of X .

Definition 2.4.9. A morphism between reduced complex analytic spaces (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) is a pair (f, f^b) , where $f : X \rightarrow Y$ is continuous, and $f^b : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a morphism of sheaves of analytic \mathbb{C} -algebras, i.e., for each $p \in X$, the map $f_p^b : \mathcal{O}_{Y, f(p)} \rightarrow \mathcal{O}_{X, p}$ is a morphism of local \mathbb{C} -algebras.

The first definition of a reduced complex analytic space has the advantage that there is a very transparent parallel with complex manifolds. As we stated previously, however, the second definition permits us to generalise to non-reduced complex analytic spaces. To show that these two definitions are equivalent, we need the following lemma.

Lemma 2.4.10. Let A, B be analytic subsets of some open sets $V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^m$ respectively. Let (A, \mathcal{O}_A) and (B, \mathcal{O}_B) denote the associated \mathbb{C} -analytic ringed spaces. If $(f, f^b) : (A, \mathcal{O}_A) \rightarrow (B, \mathcal{O}_B)$ is a morphism of \mathbb{C} -analytic ringed spaces, the continuous map $f : A \rightarrow B$ uniquely determines the map $f^b : \mathcal{O}_B \rightarrow f_*\mathcal{O}_A$.

PROOF. Let (z_1, \dots, z_n) and (w_1, \dots, w_m) denote the coordinate functions on \mathbb{C}^n and \mathbb{C}^m respectively. For each $p \in A$, the map f_p^b is a morphism of analytic \mathbb{C} -algebras, so the diagram

$$\begin{array}{ccc} \mathcal{O}_{B, f(p)} & \xrightarrow{f_p^b} & \mathcal{O}_{A, p} \\ \downarrow & & \downarrow \\ \mathbb{C} \cong \mathcal{O}_{B, f(p)} / \mathfrak{m}_{B, f(p)} & \xrightarrow{\text{Id}_{\mathbb{C}}} & \mathcal{O}_{A, p} / \mathfrak{m}_{A, p} \cong \mathbb{C} \end{array}$$

commutes. From this diagram, we see that

$$(f^b(w_k))_p = \left[f_p^b(w_k)_p \right]_{\text{mod } \mathfrak{m}_{A, p}} = \left[(w_k)_{f(p)} \right]_{\text{mod } \mathfrak{m}_{B, f(p)}} = f_k(p).$$

By *Proposition 1.2.33* f^b is uniquely determined by the images $f^b(w_k)$, $k = 1, \dots, m$, and since these images are determined by f , this completes the proof. \square

Theorem 2.4.11. The definition of a reduced complex analytic space given in *Definition 2.4.5* and the definition given in *Definition 2.4.8* are equivalent.

PROOF. Let $f : X \rightarrow Y$ be a morphism of reduced complex analytic spaces (in the sense of *Definition 2.4.6*). For any point $p \in X$, using analytic charts, the map $f_p^b : \mathcal{O}_{Y, f(p)} \rightarrow \mathcal{O}_{X, p}$ is a morphism of analytic \mathbb{C} -algebras given by $\mathcal{O}_{Y, f(p)} \ni g \mapsto g \circ f \in \mathcal{O}_{X, p}$. Conversely, let $(f, f^b) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of reduced complex analytic spaces (in the sense of *Definition 2.4.9*). We cover X by local models $(U_\lambda, \varphi_\lambda)$, $\lambda \in \Lambda$, where $(\varphi_\lambda, \varphi_\lambda^b) : (U_\lambda, \mathcal{O}_X|_{U_\lambda}) \rightarrow (A_\lambda, \mathcal{O}_{A_\lambda})$ are isomorphisms of \mathbb{C} -analytic ringed spaces. We need

to show that the transition maps $\varphi_\mu \circ \varphi_\lambda^{-1}$ are morphisms of analytic sets. By *Lemma 2.4.10* the components of $\varphi_\mu \circ \varphi_\lambda$ are given by $(\varphi_\mu^{-1})^b \circ \varphi_\lambda^b(z_k)$ for each $k \in \{1, \dots, n_\lambda\}$. Hence, these maps are holomorphic, and $(U_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ is an analytic atlas for X . It can be easily shown that the equivalence class of this atlas is independent of the chosen covering by local models. \square

We now proceed with the most general definition of a complex analytic space (possibly non-reduced).

Let $G \subseteq \mathbb{C}^n$ be a domain, and $\mathcal{J} \subset \mathcal{O}_G$ an ideal sheaf of finite type. That is, given any point $p \in G$, there is a neighbourhood U of p and a finite number of analytic functions $f_1, \dots, f_\ell \in \mathcal{O}(U)$ which generate \mathcal{J}_U . Let $V(\mathcal{J}) := \{p \in G : \mathcal{J}_p \neq \mathcal{O}_p\} = \{p \in G : \mathcal{O}_p/\mathcal{J}_p \neq 0\}$. We note that the set $V(\mathcal{J})$ is an analytic set. Indeed, for a point $p \in G$, $\mathcal{J}_p \neq \mathcal{O}_p$ if and only if $f(p) = 0$ for every $f \in \mathcal{J}_p$. Since \mathcal{J} is of finite type, there is a neighbourhood U of p such that

$$U \cap V(\mathcal{J}) = V(f_1, \dots, f_\ell) = \{p \in U : f_k(p) = 0, 1 \leq k \leq \ell\}.$$

We offer the definition of local model which fits into this new framework.

Definition 2.4.12. A *local model* for a complex analytic space (X, \mathcal{O}_X) is a \mathbb{C} -analytic ringed space $(V(\mathcal{J}), (\mathcal{O}/\mathcal{J})|_{V(\mathcal{J})})$ where $\mathcal{J} \subset \mathcal{O}$ is an ideal sheaf of finite type.

Note that if A is an analytic set in some domain $G \subseteq \mathbb{C}^n$, then each morphism $f : A \rightarrow \mathbb{C}$ locally lifts to a holomorphic function in some open set $U \subset G$. Any two holomorphic functions f, g on $A \cap U$ define the same function if and only if $(f - g)(p) = 0$ for all $p \in A \cap U$. Therefore, $\mathcal{O}_A \cong (\mathcal{O}_G/\mathcal{J}(A))|_A$, where $\mathcal{J}(A)$ is the *full ideal sheaf of A* , i.e., the sheaf which assigns to each open set U , the ideal

$$\mathcal{J}(A)(U) := \{f \in \mathcal{O}_X(U) : A \cap U \subset V(f)\}.$$

Definition 2.4.13. A *complex analytic space* is a \mathbb{C} -analytic ringed space (X, \mathcal{O}_X) , where X is a Hausdorff, second countable topological space and, for every point $x \in X$ there is an open neighbourhood $U \subseteq X$ of x such that $(U, \mathcal{O}_X|_U)$ is isomorphic (as a \mathbb{C} -analytic ringed space) to a local model $(V(\mathcal{J}), (\mathcal{O}/\mathcal{J})|_{V(\mathcal{J})})$. A *morphism of complex analytic spaces* (or a *holomorphic map*) is a morphism of \mathbb{C} -analytic ringed spaces.

For any point $p \in X$, we observe that

$$\mathcal{O}_{X,p} \cong \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}_0 \cong \mathbb{C}\{z\}/\langle f_1, \dots, f_\ell \rangle.$$

We refer to the functions f_1, \dots, f_ℓ as the *local coordinates* of X at p .

Remark 2.4.14. What distinguishes a non-reduced complex analytic space from a reduced complex analytic space is that if X is non-reduced, the structure sheaf \mathcal{O}_X is no longer a subsheaf of the sheaf \mathcal{C}_X of continuous functions. Indeed, the structure sheaf \mathcal{O}_X may have nilpotents. A particularly treacherous consequence of this fact is that the sections of \mathcal{O}_X are no longer determined by their values at points. It is therefore perhaps misleading to refer to sections of \mathcal{O}_X as functions. Nevertheless, this language is maintained and should not cause confusion.

Example 2.4.15. The prototypical example of a non-reduced complex analytic space is a *fat point of length two*³. The underlying topological space of a fat point consists only of a single point $X = \{\text{pt}\}$, and so the structure sheaf \mathcal{O}_X is determined uniquely by the stalk at this point. If we take $\mathcal{O}_{X,\text{pt}}$ to be the non-reduced ring⁴ $\mathbb{C}[z]/\langle z^2 \rangle$, then $z \in \mathcal{O}_{X,\text{pt}}$, but $z^2 = 0$ in $\mathcal{O}_{X,\text{pt}}$.

Definition 2.4.16. Let (X, \mathcal{O}_X) be a complex analytic space and $\mathcal{N}_X \subset \mathcal{O}_X$ the nilradical sheaf. We define the *reduction of X* to be the complex analytic space $(X_{\text{red}}, \text{red } \mathcal{O}_X)$, where $X_{\text{red}} := X$ and $\text{red } \mathcal{O}_X := \mathcal{O}_X / \mathcal{N}_X$. If $U \subseteq X$ is an open set, and $f \in \mathcal{O}_X(U)$, we define the *reduction of f* to be the continuous function $\text{red}(f) : U \rightarrow \mathbb{C}$, $U \ni p \mapsto f(p) := \text{eval}_p(f_p)$. The map $\text{eval}_p(f_p)$ is the *evaluation* map $\text{eval}_p := \text{proj}_{\mathbb{C}} : \mathcal{O}_{X,p} = \mathbb{C} \oplus \mathfrak{m}_p \rightarrow \mathbb{C}$, and f_p is the stalk of f at p as usual.

To see that $\text{red}(f)$ is continuous, observe that locally, we may assume that $(X, \mathcal{O}_X) = (V(\mathcal{J}), (\mathcal{O}_V/\mathcal{J})|_{V(\mathcal{J})})$ is a local model, where $V \subset \mathbb{C}^m$ is an open set containing $V(\mathcal{J})$. By shrinking V if necessary, we may lift $\text{red}(f)$ to a section $\tilde{f} \in \mathcal{O}_{\mathbb{C}^n}(V)$, i.e., lift $\text{red}(f)$ to a holomorphic function $\tilde{f} : V \rightarrow \mathbb{C}$ such that $\text{red}(f)(p) = \tilde{f}(p)$ for all $p \in V$. Since \tilde{f} is holomorphic, it is, in particular, a continuous function, and so $\text{red}(f)$ is continuous, as claimed.

Triviality 2.4.17. It is clear that a complex analytic space is reduced if and only if it coincides with its reduction.

We mention the following important (non-trivial) theorem due to K. Oka [52].

Theorem 2.4.18. (*Oka's Coherence Theorem*). The structure sheaf \mathcal{O}_X of a complex analytic space (X, \mathcal{O}_X) is a coherent sheaf.

Definition 2.4.19. Let $X := (X, \mathcal{O}_X)$ be a complex analytic space. We say that X is a *Stein space* if

- (i) X is holomorphically convex.
- (ii) X is holomorphically separable.

In more detail, we say that X is *holomorphically convex* if for every compact set $K \subset X$, the holomorphically convex hull

$$\widehat{K} = \bigcap_{f \in \mathcal{O}_X(X)} \left\{ x \in X : |f(x)| \leq \max_{z \in K} |f(z)| \right\}$$

is compact in X .

Topological Remarks. We conclude this section with some remarks on the topology of complex analytic spaces. We first consider the following definition that appears throughout many of the proofs in this thesis.

³This terminology is what is used in [24]. In [28, p. 265], the terminology *dual numbers* is used.

⁴Recall that a ring R is said to be *reduced* if it has no nilpotent elements. If R is not reduced, we say that it is non-reduced.

Definition 2.4.20. Let X be a topological space and $(K_\nu)_{\nu \geq 1}$ a sequence of compact sets in X . We say that $(K_\nu)_{\nu \geq 1}$ is an *exhaustion of X by compact sets* if

- (i) $\overset{\circ}{K}_\nu \subset K_{\nu+1}$ for all $\nu \geq 1$.
- (ii) $X = \bigcup_{\nu \geq 1} K_\nu$.

Lemma 2.4.21. Let X be a topological space that is locally compact⁵ and second countable, then X admits an exhaustion by compact sets.

PROOF. Let $\mathcal{B} := (B_\mu)_{\mu \in \mathbb{N}}$ be a countable base of open sets for X . For each $\lambda \in \mathbb{N}$, we let $U_\lambda := \bigcup_{1 \leq \mu \leq \lambda} B_\mu$. We proceed inductively to construct an exhaustion $(K_\nu)_{\nu \in \mathbb{N}}$ of X by compact sets. To this end, set $K_1 = \overline{B}_1$, and suppose that K_1, \dots, K_ν have been chosen such that $K_{\mu-1} \subset \overset{\circ}{K}_\mu$ for each $2 \leq \mu \leq \nu$. Since X is locally compact, the closure \overline{B}_μ of each B_μ is a compact set, and in particular, \overline{U}_λ is compact for each $\lambda \in \mathbb{N}$. We then choose λ to be the smallest positive integer such that $K_\nu \subset U_\lambda$, then $K_{\nu+1} := \overline{U}_\lambda$ is compact, and $K_\nu \subset \overset{\circ}{K}_{\nu+1}$, as required. \square

Proposition 2.4.22. Every complex analytic space X is locally compact and second-countable. In particular, every complex analytic space is paracompact and admits an exhaustion by compact sets.

PROOF. We have assumed that all complex analytic spaces are second countable. To see that X is locally compact, consider that for any point $p \in X$, we have a \mathbb{C} -analytic ringed space $(U_\lambda, \mathcal{O}_{U_\lambda})$ such that $p \in U_\lambda$ and $(U_\lambda, \mathcal{O}_{U_\lambda})$ is isomorphic (as a \mathbb{C} -analytic ringed space) to a local model $(V(\mathcal{J}_\lambda), (\mathcal{O}_G/\mathcal{J}_\lambda)|_{V(\mathcal{J}_\lambda)})$. Since $V(\mathcal{J}_\lambda) \subset \mathbb{C}^{n_\lambda}$, and \mathbb{C}^{n_λ} is locally compact, X is locally compact as claimed. Paracompactness is therefore immediate, and by applying *Lemma 2.4.21*, we see that every complex analytic space admits an exhaustion by compact sets. \square

The following trivial fact is what makes exhaustions by compact sets very useful to us.

Triviality 2.4.23. Let $X := (X, \mathcal{O}_X)$ be a complex analytic space, and let $(K_\nu)_{\nu \geq 1}$ be an exhaustion of X by compact sets. Let \mathcal{S} be a sheaf on X , and $s_\nu \in \mathcal{S}(K_\nu)$ a sequence of sections such that, for each $\nu \geq 1$, $s_{\nu+1}|_{K_\nu} = s_\nu$. Then there exists a unique global section $s \in \mathcal{S}(X)$ such that, for each $\nu \geq 1$, $s|_{K_\nu} = s_\nu$.

⁵By a topological space X being locally compact, we mean that X is Hausdorff and every point $x \in X$ has a neighbourhood which is compact.

CHAPTER 3

Sheaf Cohomology and Cartan's Theorem B

§3.1. SHEAF COHOMOLOGY – THE STATEMENTS

The purpose of this section is to survey some results on sheaf cohomology that will be essential to the proof of Cartan's theorem B. Readers familiar with sheaf cohomology may wish to skip this section and refer back only when necessary. Throughout our discussion of sheaf cohomology via flabby and soft sheaves and Čech cohomology, X will denote a topological space, and \mathcal{R} will be a sheaf of rings on X .

Sheaf Cohomology Via Flabby and Soft Sheaves. Suppose that we have an exact sequence of sheaves of \mathcal{R} -modules

$$0 \longrightarrow \mathcal{S} \xrightarrow{\iota} \mathcal{S}^0 \xrightarrow{d_0} \mathcal{S}^1 \xrightarrow{d_1} \dots \xrightarrow{d_{q-1}} \mathcal{S}^q \longrightarrow \dots,$$

over X . We have an induced sequence at the level of global sections

$$0 \longrightarrow \mathcal{S}(X) \xrightarrow{\iota^*} \mathcal{S}^0(X) \xrightarrow{d_0^*} \mathcal{S}^1(X) \xrightarrow{d_1^*} \dots \xrightarrow{d_{q-1}^*} \mathcal{S}^q(X) \longrightarrow \dots, \quad (15)$$

which may fail to be exact. Since $d_q^* \circ d_{q-1}^* = 0$ for all $q \geq 1$ however, (15) forms a complex of $\mathcal{R}(X)$ -modules. The failure of an exact sequence of sheaves to induce an exact sequence at the level of global sections is measured by sheaf cohomology (whatever this is). The following class of sheaves will be used to compute the sheaf cohomology groups (see *Definition 3.1.7*).

Notational Remark 3.1.1. In the interests of clarity, we will write d_q for the coboundary map at the level of global sections in place of d_q^* .

Definition 3.1.2. A sheaf of \mathcal{R} -modules \mathcal{S} on X is said to be *flabby* (or *flasque*) if for every open set $U \subseteq X$, the restriction map $\text{res}_U^X : \mathcal{S}(X) \longrightarrow \mathcal{S}(U)$ is surjective.

Lemma 3.1.3. ([19, p. 26]) Let

$$0 \longrightarrow \mathcal{S}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}'' \longrightarrow 0$$

be an exact sequence of sheaves of \mathcal{R} -modules on X . If \mathcal{S}' is flabby, then the sequence

$$0 \longrightarrow \mathcal{S}'(X) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{S}''(X) \longrightarrow 0$$

is exact. Moreover, if \mathcal{S}' and \mathcal{S} are flabby sheaves, then \mathcal{S}'' is also flabby.

We may strengthen *Lemma 3.1.3* to establish the following result:

Proposition 3.1.4. ([19, p. 26]). Let

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}^0 \longrightarrow \mathcal{S}^1 \longrightarrow \cdots \longrightarrow \mathcal{S}^q \longrightarrow \cdots$$

be an exact sequence of sheaves of \mathcal{R} -modules over X . If the sheaves $\mathcal{S}, \mathcal{S}^q, q \geq 0$, are all flabby, then the induced map at the level of global sections

$$0 \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{S}^0(X) \longrightarrow \mathcal{S}^1(X) \longrightarrow \cdots \longrightarrow \mathcal{S}^q(X) \longrightarrow \cdots$$

is exact.

Definition 3.1.5. Let \mathcal{S} be a sheaf of \mathcal{R} -modules on X . We may associate to \mathcal{S} a flabby sheaf $\mathcal{F}(\mathcal{S})$ called the *flabby sheaf associated to \mathcal{S}* . This sheaf is defined by the assignment

$$U \longmapsto \mathcal{F}(\mathcal{S})(U) := \left\{ s : U \longrightarrow \bigcup_{p \in U} \mathcal{S}_p : s(p) \in \mathcal{S}_p, \quad \forall p \in U \right\},$$

with the restriction maps defined in the obvious way.

There is a natural inclusion $\iota : \mathcal{S} \longrightarrow \mathcal{F}(\mathcal{S})$ and for any morphism $\varphi : \mathcal{S} \longrightarrow \mathcal{T}$, there is an induced map $\mathcal{F}(\varphi) : \mathcal{F}(\mathcal{S}) \longrightarrow \mathcal{F}(\mathcal{T})$. The assignment of $\mathcal{S} \rightsquigarrow \mathcal{F}(\mathcal{S})$ defines an exact functor called the *flabby functor*.

Construction 3.1.6. Let \mathcal{S} be a sheaf of \mathcal{R} -modules on X . We construct an exact sequence of \mathcal{R} -modules for \mathcal{S} in the following way:

- (i) Set $\mathcal{S}^0 := \mathcal{F}(\mathcal{S})$, and let $\iota : \mathcal{S} \longrightarrow \mathcal{S}^0$ denote the inclusion map. We represent this by an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}^0.$$

- (ii) Set $\mathcal{S}^1 := \mathcal{F}(\mathcal{S}^0/\iota(\mathcal{S}))$, and define $d_0 : \mathcal{S}^0 \longrightarrow \mathcal{S}^1$ to be the composition of the quotient map $\mathcal{S}^0 \longrightarrow \mathcal{S}^0/\iota(\mathcal{S})$ and the inclusion map $\mathcal{S}^0/\iota(\mathcal{S}) \hookrightarrow \mathcal{S}^1$. We therefore have an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}^0 \longrightarrow \mathcal{S}^1.$$

- (iii) We iterate this process to obtain an exact sequence

$$0 \longrightarrow \mathcal{S} \xrightarrow{\iota} \mathcal{S}^0 \xrightarrow{d_0} \mathcal{S}^1 \xrightarrow{d_1} \mathcal{S}^2 \longrightarrow \cdots,$$

where $\mathcal{S}^q := \mathcal{F}(\mathcal{S}^{q-1}/\mathcal{I}m(d_{q-1}))$ is flabby for each $q \geq 0$, and the map $d_q : \mathcal{S}^q \longrightarrow \mathcal{S}^{q+1}$ is given by the composition of the quotient $\mathcal{S}^q \longrightarrow \mathcal{F}(\mathcal{S}^q/\mathcal{I}m(d_{q-1}))$ and the inclusion map $\mathcal{S}^q/\mathcal{I}m(d_{q-1}) \hookrightarrow \mathcal{S}^{q+1}$.

Therefore, associated to any sheaf \mathcal{S} , we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{S} \xrightarrow{\iota} \mathcal{S}^0 \xrightarrow{d_0} \mathcal{S}^1 \xrightarrow{d_1} \mathcal{S}^2 \longrightarrow \cdots \quad (16)$$

which induces an exact sequence at the level of global sections

$$\mathcal{S}(X) \xrightarrow{\iota} \mathcal{S}^0(X) \xrightarrow{d_0} \mathcal{S}^1(X) \xrightarrow{d_1} \mathcal{S}^2(X) \longrightarrow \cdots \quad (17)$$

The exact sequence (16) is called a *flabby resolution* of \mathcal{S} , and is denoted by \mathcal{S}^\bullet . Moreover, since $d_q \circ d_{q-1} = 0$ for each $q \geq 1$, the exact sequence (17) forms a complex of $\mathcal{R}(X)$ -modules which we denote by $\mathcal{S}^\bullet(X)$.

Definition 3.1.7. Let \mathcal{S} be a sheaf of \mathcal{R} -modules on X . The sheaf cohomology groups of \mathcal{S} are the $\mathcal{R}(X)$ -modules

$$H^0(X, \mathcal{S}) := \mathcal{S}(X), \quad \text{and} \quad H^q(X, \mathcal{S}) := \frac{\ker(d_q)}{\text{im}(d_{q-1})}, \quad \forall q \geq 1.$$

Theorem 3.1.8. ([19, p. 30]). Let $0 \rightarrow \mathcal{S}' \rightarrow \mathcal{S} \rightarrow \mathcal{S}'' \rightarrow 0$ be an exact sequence of sheaves of \mathcal{R} -modules over X . For each $q \geq 0$ there is a connecting homomorphism $\delta^q : H^q(X, \mathcal{S}'') \rightarrow H^{q+1}(X, \mathcal{S}')$ such that the sequence

$$\cdots \rightarrow H^q(X, \mathcal{S}') \rightarrow H^q(X, \mathcal{S}) \rightarrow H^q(X, \mathcal{S}'') \xrightarrow{\delta^q} H^{q+1}(X, \mathcal{S}') \rightarrow \cdots$$

is exact.

We have the following consequence of *Proposition 3.1.4*.

Proposition 3.1.9. ([19, p. 31]). Let \mathcal{S} be a sheaf of \mathcal{R} -modules on X . If \mathcal{S} is flabby then $H^q(X, \mathcal{S}) = 0$ for all $q \geq 1$.

Definition 3.1.10. An exact sequence of sheaves of \mathcal{R} -modules

$$0 \rightarrow \mathcal{S} \xrightarrow{\iota} \mathcal{S}^0 \xrightarrow{d_0} \mathcal{S}^1 \xrightarrow{d_1} \mathcal{S}^2 \rightarrow \cdots$$

over X , is called an *acyclic resolution* if $H^q(X, \mathcal{S}^k) = 0$ for all $k \geq 0$, and all $q \geq 1$.

By *Proposition 3.1.9* a resolution of flabby sheaves is an acyclic resolution.

Theorem 3.1.11. (*The Formal de Rham Lemma*, [19, p. 32]). Let

$$0 \rightarrow \mathcal{S} \xrightarrow{\iota} \mathcal{S}^0 \xrightarrow{d_0} \mathcal{S}^1 \xrightarrow{d_1} \mathcal{S}^2 \rightarrow \cdots$$

be an acyclic resolution of \mathcal{S} , and let $\mathcal{S}^\bullet(X)$ denote the induced complex at the level of sections. Then for all $q \geq 0$, there exist natural $\mathcal{R}(X)$ -module isomorphisms

$$H^q(\mathcal{S}^\bullet(X)) \cong H^q(X, \mathcal{S}).$$

We consider the following class of sheaves which allow us to extend sections over open sets to global sections, c.f., *Definition 3.1.2*.

Definition 3.1.12. A sheaf of \mathcal{R} -modules \mathcal{S} on X is said to be *soft* if for every closed set $K \subset X$, the restriction map $\text{res}_K^X : \mathcal{S}(X) \rightarrow \mathcal{S}(K)$ is surjective.

We have the following analogues of *Lemma 3.1.3* and *Proposition 3.1.4*:

Lemma 3.1.13. ([19, p. 26]). Assume that X is paracompact, and suppose that

$$0 \longrightarrow \mathcal{S}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}'' \longrightarrow 0$$

is an exact sequence of sheaves of \mathcal{R} -modules on X . If \mathcal{S}' is soft, then the associated sequence at the level of global sections

$$0 \longrightarrow \mathcal{S}'(X) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{S}''(X) \longrightarrow 0$$

is exact. Moreover, if \mathcal{S}' and \mathcal{S} are soft, then \mathcal{S}'' is also soft.

Proposition 3.1.14. ([19, p. 26]). Let

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}^0 \xrightarrow{d_0} \mathcal{S}^1 \xrightarrow{d_1} \dots \xrightarrow{d_{q-1}} \mathcal{S}^q \xrightarrow{d_q} \dots$$

be an exact sequence of sheaves of \mathcal{R} -modules on a paracompact topological space X . If every sheaf \mathcal{S}^q , $q \geq 0$, is soft, then the associated sequence at the level of global sections

$$\mathcal{S}^0(X) \xrightarrow{d_0} \mathcal{S}^1(X) \xrightarrow{d_1} \dots \xrightarrow{d_{q-1}} \mathcal{S}^q(X) \xrightarrow{d_q} \dots$$

is exact.

The following important consequence of point-set topology is used in the proof of *Proposition 3.1.4*.

Proposition 3.1.15. Suppose that X is paracompact, then every flabby sheaf on X is soft. In particular, since every complex analytic space is locally compact and, by assumption, second countable, every flabby sheaf on a complex analytic space is soft.

Proposition 3.1.16. If \mathcal{S} is a soft sheaf on X , then $H^q(X, \mathcal{S}) = 0$ for all $q \geq 1$.

Definition 3.1.17. Assume that X be a paracompact, and suppose that \mathcal{S} is a sheaf of \mathcal{R} -modules on X . An exact sequence of form

$$0 \longrightarrow \mathcal{S} \xrightarrow{i} \mathcal{S}^0 \xrightarrow{i} \mathcal{S}^1 \xrightarrow{d_1} \mathcal{S}^2 \xrightarrow{d_2} \dots,$$

where $d_{q+1} \circ d_q = 0$, and \mathcal{S}^q is soft for each $q \geq 0$, is called a *soft resolution* of \mathcal{S} .

Let \mathcal{S} be a sheaf on a paracompact topological space X . By *Proposition 3.1.15*, every flabby resolution of \mathcal{S} is a soft resolution. Moreover, by *Proposition 3.1.16* every soft resolution is acyclic. Hence, by *Theorem 3.1.11*, the cohomology groups of X computed via a resolution of flabby sheaves are naturally isomorphic to the cohomology groups computed via a resolution of soft sheaves.

Čech Cohomology. Let $\mathfrak{U} := (U_\lambda)_{\lambda \in \Lambda}$ be an open cover of X . For any $(k+1)$ -tuple $(\lambda_0, \dots, \lambda_k) \in I^{k+1}$, for some indexing set I , we set $U_{\lambda_0 \dots \lambda_k} := U_{\lambda_0} \cap \dots \cap U_{\lambda_k}$, and for each $k \geq 0$, we define an $\mathcal{R}(X)$ -module:

$$C^k(\mathfrak{U}, \mathcal{S}) := \prod_{\lambda_0 < \dots < \lambda_k} \mathcal{S}(U_{\lambda_0 \dots \lambda_k}).$$

An element $\alpha \in C^k(\mathfrak{U}, \mathcal{S})$ is a function which assigns to each $(k+1)$ -tuple $(\lambda_0, \dots, \lambda_k)$, an element $\alpha_{\lambda_0 \dots \lambda_k}$ in $\mathcal{S}(U_{\lambda_0 \dots \lambda_k})$, we call α a *k-cochain*.

We define a coboundary map $d_k : C^k(\mathfrak{U}, \mathcal{S}) \rightarrow C^{k+1}(\mathfrak{U}, \mathcal{S})$ by

$$(\delta_k \alpha)_{\lambda_0 \dots \lambda_{k+1}} := \sum_{j=0}^{k+1} (-1)^j \alpha_{\lambda_0 \dots \widehat{\lambda}_j \dots \lambda_{k+1}} \Big|_{U_{\lambda_0 \dots \widehat{\lambda}_j \dots \lambda_{k+1}}},$$

where $\widehat{\lambda}_j$ means that we omit λ_j . The reader may easily verify by direct computation that $\delta_{k+1} \circ \delta_k = 0$, and $C^\bullet(\mathfrak{U}, \mathcal{S}) := (C^k(\mathfrak{U}, \mathcal{S}), d_k)$ is a complex of $\mathcal{R}(X)$ -modules.

Definition 3.1.18. The Čech cohomology modules of the sheaf \mathcal{S} over X , with respect to the open cover \mathfrak{U} , are defined by

$$\check{H}^0(\mathfrak{U}, \mathcal{S}) := \ker(d_0), \quad \text{and} \quad \check{H}^q(\mathfrak{U}, \mathcal{S}) := \frac{\ker(d_q)}{\text{im}(d_{q-1})}, \quad \forall q \geq 1.$$

Now suppose that $\mathfrak{V} := (V_\mu)_{\mu \in \Lambda'}$ is a refinement of \mathfrak{U} , we will write $\mathfrak{V} < \mathfrak{U}$ to signify this, and let $\gamma : \Lambda' \rightarrow \Lambda$ be a refinement mapping. We have a natural $\mathcal{R}(X)$ -module homomorphism $C^k(\gamma) : C^k(\mathfrak{U}, \mathcal{S}) \rightarrow C^k(\mathfrak{V}, \mathcal{S})$, where $C^k(\alpha_{\mu_0 \dots \mu_k}) := \alpha_{\gamma(\lambda_0) \dots \gamma(\lambda_k)}|_{V_{\mu_0 \dots \mu_k}}$. These maps are compatible with the coboundary maps, and for each $k \geq 0$, we have induced maps $h^k(\gamma) : \check{H}^k(\mathfrak{U}, \mathcal{S}) \rightarrow \check{H}^k(\mathfrak{V}, \mathcal{S})$ on the cohomology modules. One may show that these maps $h^k(\gamma)$ are independent of the choice of refinement map γ , and hence, we write $h^k(\mathfrak{U}, \mathfrak{V}) := h^k(\gamma)$. Then, for each $k \geq 0$, we obtain a directed system $(\check{H}^k(\mathfrak{U}, \mathcal{S}), h^k(\mathfrak{U}, \mathfrak{V}))$, which is directed with respect to the relation $\mathfrak{V} < \mathfrak{U}$.

Definition 3.1.19. The Čech cohomology groups of a sheaf \mathcal{S} over X are given by the inductive limit

$$\check{H}^q(X, \mathcal{S}) := \varinjlim \check{H}^q(\mathfrak{U}, \mathcal{S}).$$

Proposition 3.1.20. ([19, p. 34]). The functors $\mathcal{S} \rightsquigarrow \mathcal{S}(X)$ and $\mathcal{S} \rightsquigarrow \check{H}^0(X, \mathcal{S})$ are isomorphic.

Theorem 3.1.21. ([19, p. 34]). Assume that X is paracompact. For an exact sequence

$$0 \rightarrow \mathcal{S}' \rightarrow \mathcal{S} \rightarrow \mathcal{S}'' \rightarrow 0$$

of \mathcal{R} -modules, there is an induced exact sequence of Čech cohomology modules

$$\dots \rightarrow \check{H}^q(X, \mathcal{S}) \rightarrow \check{H}^q(X, \mathcal{S}'') \xrightarrow{\check{\delta}_q} \check{H}^{q+1}(X, \mathcal{S}') \rightarrow \check{H}^{q+1}(X, \mathcal{S}) \rightarrow \dots$$

Definition 3.1.22. We say that an open cover $\mathfrak{U} = (U_\lambda)_{\lambda \in \Lambda}$ of X is *acyclic* with respect to the sheaf of \mathcal{R} -modules \mathcal{S} if for all $k \geq 0$, and $q \geq 1$, we have $H^q(U_{\lambda_0 \dots \lambda_k}, \mathcal{S}) = 0$.

Theorem 3.1.23. (*The Leray Theorem*, [19, p. 43]). If \mathfrak{U} is a locally finite cover of X which is acyclic with respect to the sheaf \mathcal{S} , then there is a natural $\mathcal{R}(X)$ -isomorphism

$$\check{H}^q(\mathfrak{U}, \mathcal{S}) \cong H^q(X, \mathcal{S})$$

for all $q \geq 0$.

Note that if \mathcal{S} is a flabby sheaf, then every cover \mathfrak{U} of X is acyclic with respect to \mathcal{S} . Hence, by *Theorem 3.1.23*, if \mathfrak{U} is locally finite, then $\check{H}^q(\mathfrak{U}, \mathcal{S}) = 0$ for each $q \geq 1$. In particular, for a complex analytic space, which is paracompact, every open cover has a locally finite refinement. We, therefore, obtain the following corollary.

Corollary 3.1.24. ([19, p. 43]). If \mathcal{S} is a flabby sheaf on a complex analytic space (X, \mathcal{O}_X) , then $\check{H}^q(\mathfrak{U}, \mathcal{S}) = 0$, for all $q \geq 1$.

The above corollary is useful in establishing uniqueness results for cohomology theories. Let us mention a very important theorem on the uniqueness of sheaf cohomology.

Theorem 3.1.25. ([19, p. 31]). For each $q \geq 0$, let \tilde{H}^q be a sequence of functors whose associated connecting homomorphisms are denoted by $\tilde{\delta}^q$. Assume that these functors and homomorphisms satisfy:

- (i) $\tilde{H}^0(X, \mathcal{S}) = \mathcal{S}(X)$ for any sheaf of \mathcal{R} -modules \mathcal{S} .
- (ii) For any exact sequence of sheaves of \mathcal{R} -modules

$$0 \longrightarrow \mathcal{S}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}'' \longrightarrow 0,$$

the connecting homomorphisms $\tilde{\delta}^q$ induce a long exact exact sequence

$$0 \longrightarrow \tilde{H}^0(X, \mathcal{S}) \longrightarrow \dots \longrightarrow \tilde{H}^q(X, \mathcal{S}) \longrightarrow \tilde{H}^q(X, \mathcal{S}'') \xrightarrow{\tilde{\delta}^q} \tilde{H}^{q+1}(X, \mathcal{S}') \longrightarrow \dots.$$

- (iii) If \mathcal{S} is a flabby sheaf of \mathcal{R} -modules on X , then $\tilde{H}^q(X, \mathcal{S}) = 0$ for all $q \geq 1$.

Then for every $q \geq 0$, there is a natural functor isomorphism

$$\Phi^q : H^q(X, \mathcal{S}) \longrightarrow \tilde{H}^q(X, \mathcal{S})$$

which is compatible with the connecting homomorphisms.

Combining *Definition 3.1.19*, *Proposition 3.1.20*, *Theorem 3.1.21*, *Corollary 3.1.24*, and *Theorem 3.1.25*, we arrive at the following theorem.

Theorem 3.1.26. ([19, p. 43]). Let \mathcal{S} be a sheaf on X , where X is paracompact. Then for all $q \geq 0$,

$$\check{H}^q(X, \mathcal{S}) \cong H^q(X, \mathcal{S}).$$

Cohomology of Finite Maps. We conclude this section with a discussion of finite maps. Such maps will play a very important role in the proof of Cartan's Theorem B.

Definition 3.1.27. Let $f : X \rightarrow Y$ be a continuous map between locally compact topological spaces.

- We say that $f : X \rightarrow Y$ is *proper* if for every compact set $K \subset Y$, the preimage $f^{-1}(K)$ is compact.
- We say that f is a *closed map* if $f(K) \subseteq Y$ is closed for every closed set $K \subseteq X$. If, in addition to this, f is continuous, and for every $y \in Y$, the fibre $f^{-1}(y)$ is a finite set, then f is said to be a *finite map*.

Note that if we write that $f : X \rightarrow Y$ is a finite map, it is understood that X and Y are both locally compact topological spaces. Further, we remind the reader that all locally compact topological spaces are assumed to be Hausdorff.

Example 3.1.28. The map $f : \mathbb{C} \rightarrow \mathbb{C}$, defined by $f(z) := z^n$ is finite for each $n \in \mathbb{N}$. The projection map $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$, defined by $\pi(z, w) = z$, is not finite however, since $\pi^{-1}(0) = \{0\} \times \mathbb{C}$.

Definition 3.1.29. Let \mathcal{S} be a sheaf of abelian groups on X . For a continuous map $f : X \rightarrow Y$, we define the *direct image sheaf* $f_*\mathcal{S}$ on Y to be the sheaf associated to the presheaf

$$Y \supset V \mapsto \mathcal{S}(f^{-1}(V)) =: (f_*\mathcal{S})(V).$$

Proposition 3.1.30. ([19, p. 47]). Let $f : X \rightarrow Y$ be a finite map, and suppose that over X we have an exact sequence of sheaves of abelian groups

$$\mathcal{R} \xrightarrow{\varphi} \mathcal{S} \xrightarrow{\psi} \mathcal{T}.$$

Then the sequence

$$f_*\mathcal{R} \xrightarrow{f_*(\varphi)} f_*\mathcal{S} \xrightarrow{f_*(\psi)} f_*\mathcal{T}$$

is also exact.

This yields the following very important theorem:

Theorem 3.1.31. ([19, p. 47]). Let $f : X \rightarrow Y$ be a finite map, and let \mathcal{S} be a sheaf of complex vector spaces over X . Then for each $q \geq 0$, we have natural complex vector space isomorphisms

$$H^q(X, \mathcal{S}) \cong H^q(Y, f_*(\mathcal{S})).$$

The above theorem is the primary motivation for the consideration of analytic blocks in place of analytic stones in §3.3. We conclude with a statement of Grauert's direct image theorem for finite maps:

Theorem 3.1.32. ([19, p. 54]). Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) denote complex analytic spaces. Let $f : X \rightarrow Y$ be a finite map, and suppose that \mathcal{S} is a coherent analytic sheaf on X . Then the direct image $f_*\mathcal{S}$ is a coherent analytic sheaf on Y .

Remark 3.1.33. Note that Grauert's direct image theorem holds also for proper maps. This generalisation however is substantially more difficult to prove (see, e.g., [20, Chapter X]).

§3.2. DOLBEAULT COHOMOLOGY

In §1.3 we discussed the calculus of differential forms and in particular, we defined the Dolbeault operator $\bar{\partial} : \Omega^{p,q}(G) \rightarrow \Omega^{p,q+1}(G)$. This differential operator will be used as a coboundary operator in Dolbeault cohomology in the same way that the exterior derivative is used in the de Rham cohomology of smooth manifolds.

Definition 3.2.1. Let $G \subseteq \mathbb{C}^n$ be a domain and $\omega \in \Omega^{p,q}(G)$ a smooth (p, q) -form. We will say that ω is $\bar{\partial}$ -closed if $\bar{\partial}\omega = 0$. If there exists a smooth $(p, q-1)$ -form $\eta \in \Omega^{p,q-1}(G)$ such that $\omega = \bar{\partial}\eta$, we say that ω is $\bar{\partial}$ -exact. The set of all smooth $\bar{\partial}$ -closed (p, q) -forms and $\bar{\partial}$ -exact (p, q) -forms will be denoted by $\mathcal{Z}^{p,q}(G)$ and $\mathcal{B}^{p,q}(G)$, respectively.

Recall from *Proposition 1.3.19* that $\bar{\partial}^2 = \bar{\partial} \circ \bar{\partial} = 0$, and therefore every $\bar{\partial}$ -exact form is $\bar{\partial}$ -closed. The converse, however, is not true in general, and the obstruction of every $\bar{\partial}$ -closed form being $\bar{\partial}$ -exact is measured by the Dolbeault cohomology groups. To discuss this cohomology theory, and relate it to the sheaf cohomology treated in §3.1, we introduce the following class of sheaves:

Definition 3.2.2. Let \mathcal{S} be a sheaf of abelian groups on a domain $G \subseteq \mathbb{C}^n$, and let $\mathfrak{U} = (U_\lambda)_{\lambda \in \Lambda}$ be a locally finite open cover of G . A *partition of unity* of \mathcal{S} , subordinate to the cover \mathfrak{U} , is a collection of morphism of sheaves $\eta_\lambda : \mathcal{S} \rightarrow \mathcal{S}$ such that

- (i) $\eta_\lambda \equiv 0$ in an open neighbourhood of $G \setminus U_\lambda$.
- (ii) $\sum_{\lambda \in \Lambda} \eta_\lambda = \text{Id}$.

Definition 3.2.3. A sheaf of abelian groups \mathcal{S} on a domain in \mathbb{C}^n is said to be *fine* if it admits a partition of unity subordinate to any locally finite open cover.

Proposition 3.2.4. ([26, p. 175]). Let \mathcal{S} be a fine sheaf on a domain in \mathbb{C}^n , then \mathcal{S} is a soft sheaf.

Lemma 3.2.5. ([26, p. 184]). If $G \subseteq \mathbb{C}^n$ is a domain, the sheaf $\mathcal{A}^{p,q}$ of smooth (p, q) -forms on G is a fine sheaf. In particular, by *Proposition 3.2.4*, $\mathcal{A}^{p,q}$ is soft.

Lemma 3.2.6. ([26, p. 184]). Let G be a domain in \mathbb{C}^n . We have an exact sequence of sheaves of abelian groups

$$0 \rightarrow \mathcal{O}_G \xrightarrow{\iota} \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}_0} \mathcal{A}^{0,1} \xrightarrow{\bar{\partial}_1} \mathcal{A}^{0,2} \rightarrow \dots \rightarrow \mathcal{A}^{0,n} \xrightarrow{\bar{\partial}_n} 0 \quad (18)$$

over G . We call the exact sequence (18) a resolution of soft sheaves.

Observe that by combining *Lemma 3.2.6*, *Proposition 3.2.4*, and *Proposition 3.1.15*, we have an exact sequence

$$0 \rightarrow \mathcal{O}_G(G) \xrightarrow{\iota} \Omega^{0,0}(G) \xrightarrow{\bar{\partial}_0} \Omega^{0,1}(G) \xrightarrow{\bar{\partial}_1} \Omega^{0,2}(G) \rightarrow \dots \rightarrow \Omega^{0,n}(G) \rightarrow 0. \quad (19)$$

The exact sequence (18) allows us to compute the Dolbeault cohomology groups of a domain $G \subseteq \mathbb{C}^n$. As the reader may already expect, the Dolbeault cohomology groups are the sheaf cohomology groups computed via this resolution of soft sheaves.

Definition 3.2.7. Let $G \subseteq \mathbb{C}^n$ be a domain in \mathbb{C}^n . The *Dolbeault cohomology groups* $\mathcal{D}^{p,q}(G)$ are the quotient groups

$$\mathcal{D}^{0,0}(G) := \mathcal{O}_G(G), \quad \text{and} \quad \mathcal{D}^{p,q}(G) := \frac{\ker(\bar{\partial}_q)}{\text{im}(\bar{\partial}_{q-1})} = \frac{\mathcal{L}^{p,q}(G)}{\mathcal{B}^{p,q}(G)}, \quad \forall q \geq 1,$$

associated to (19).

Theorem 3.2.8. ([26, p. 184]). Let G be a domain in \mathbb{C}^n . Then for each $q \geq 0$, we have isomorphisms

$$H^q(G, \mathcal{O}) \cong \mathcal{D}^{0,q}(G).$$

In what remains of this section, we want to prove that if $G \subseteq \mathbb{C}^n$ is a simply-connected polydomain (*Definition 3.2.12*), then $\mathcal{D}^{p,q}(G) = 0$ for all $p \geq 0$, and all $q \geq 1$.

Lemma 3.2.9. Let $\Delta \subset \Delta'$ denote polydisks centred at the origin in \mathbb{C}^n with $\bar{\Delta} \subset \Delta'$. Suppose that f is a smooth function in Δ' , which is holomorphic in the variables z_{k+1}, \dots, z_n , for some $1 \leq k \leq n-1$. Then there exists a function $u \in \mathcal{C}^\infty(\Delta')$, holomorphic in z_{k+1}, \dots, z_n , such that

$$\frac{\partial u}{\partial \bar{z}_k}(z) = f(z), \quad \forall z \in \Delta.$$

PROOF. Write $\Delta = \Delta_1 \times \dots \times \Delta_n$, and $\Delta' = \Delta'_1 \times \dots \times \Delta'_n$, for the product decomposition of the polydisks, and let $\rho : \mathbb{C} \rightarrow \mathbb{R}$ be a smooth bump function¹ such that

$$\rho(z) := \begin{cases} 1, & \forall z \in \Delta_k, \\ 0, & \forall z \in \mathbb{C} \setminus \Delta'_k. \end{cases}$$

We define a function u by setting

$$u(z_1, \dots, z_n) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\rho(\zeta) g(z_1, \dots, z_{k-1}, \zeta, z_{k+1}, \dots, z_n)}{\zeta - z_k} d\zeta \wedge d\bar{\zeta}.$$

This function is smooth with compact support in Δ' . Hence, by differentiating under the integral sign, we see that

$$\frac{\partial u}{\partial \bar{z}_k}(z) = (\rho \cdot g)(z) = g(z), \quad \forall z \in \Delta.$$

Further, since $\frac{\partial f}{\partial \bar{z}_\ell} = 0$, by again differentiating under the integral sign, we see that $\frac{\partial u}{\partial \bar{z}_\ell} = 0$ for each $\ell = k+1, \dots, n$. □

¹Recall that a *bump function* is a smooth \mathbb{R} -valued function that is equal to 1 on a specified closed set and is supported in a specified open set (see, e.g., [40, p. 51]). The existence of such a function is proved in [26, p. 288].

Theorem 3.2.10. (*Dolbeault Lemma*). Let $\Delta \subset \Delta'$ be two polydisks centred at the origin in \mathbb{C}^n , with $\bar{\Delta} \subset \Delta'$. Let $\omega \in \Omega^{p,q}(\Delta')$ be a smooth $\bar{\partial}$ -closed form, where $q > 0$. There exists a smooth form $\eta \in \Omega^{p,q-1}(\Delta)$ such that $\omega|_{\Delta} = \bar{\partial}\eta$.

PROOF. Without loss of generality, we may assume that $p = 0$. The form ω in the coordinates $z = (z_1, \dots, z_n)$ of \mathbb{C}^n is written

$$\omega = \sum_{1 \leq j_1 < \dots < j_q \leq n} \omega_{j_1 \dots j_q} d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q} = \sum_J \omega_J d\bar{z}_J, \quad (20)$$

where the functions ω_J are smooth in Δ' , and $J = (j_1, \dots, j_q)$. Let $0 \leq k \leq n$ denote the smallest integer such that (20) does not involve the differentials $d\bar{z}_{k+1}, \dots, d\bar{z}_n$. We proceed by induction on k . Observe that for $k = 0$, since $q > 0$, there is nothing to prove. Therefore, assume $k \geq 1$, and that the result has been shown for $k - 1$. Choose smooth forms $\alpha \in \Omega^{0,q-1}(\Delta')$, and $\beta \in \Omega^{0,q}(\Delta')$, not involving the differentials $d\bar{z}_k, \dots, d\bar{z}_n$, such that

$$\omega = d\bar{z}_k \wedge \alpha + \beta.$$

We will signify that $d\bar{z}_k, \dots, d\bar{z}_n$ does not occur in a summation by writing $'\sum$. Now, if $\alpha = '\sum_{|J|=q-1} \alpha_J d\bar{z}_J$, then

$$\begin{aligned} 0 = \bar{\partial}\omega &= -d\bar{z}_k \wedge \bar{\partial}\alpha + \bar{\partial}\beta \\ &= -\sum_{\ell \neq k} d\bar{z}_k \wedge d\bar{z}_\ell \wedge \left('\sum_{|J|=q-1} \frac{\partial \alpha_J}{\partial \bar{z}_\ell} d\bar{z}_J \right) + \bar{\partial}\beta. \end{aligned}$$

Therefore, $\frac{\partial \alpha_J}{\partial \bar{z}_\ell} \equiv 0$ for all $\ell > k$, and by *Lemma 3.1.3* we can choose smooth functions $A_J \in \mathcal{C}^\infty(\Delta')$, holomorphic in z_{k+1}, \dots, z_n , such that $\frac{\partial A_J}{\partial \bar{z}_k} = \alpha_J$ on Δ . Set $\gamma := '\sum_{|J|=q-1} A_J d\bar{z}_J$. Then

$$\begin{aligned} \bar{\partial}\gamma &= '\sum_{|J|=q-1} \sum_{\ell \notin J} \frac{\partial A_J}{\partial \bar{z}_\ell} d\bar{z}_\ell \wedge d\bar{z}_J \\ &= '\sum_{|J|=q-1} \left(\alpha_J d\bar{z}_k \wedge d\bar{z}_J + \sum_{\ell < k, \ell \notin J} \frac{\partial A_J}{\partial \bar{z}_\ell} d\bar{z}_\ell \wedge d\bar{z}_J \right) \\ &= d\bar{z}_k \wedge '\sum_{|J|=q-1} \alpha_J d\bar{z}_J + \dots \\ &= d\bar{z}_k \wedge \alpha + \xi, \end{aligned}$$

where ξ is a smooth form not involving $d\bar{z}_k, \dots, d\bar{z}_n$. We note that $\beta - \xi$ does not involve the differentials $d\bar{z}_k, \dots, d\bar{z}_n$, and since $\beta - \xi = (\omega - d\bar{z}_k \wedge \alpha) - (\bar{\partial}\gamma - d\bar{z}_k \wedge \alpha) = \omega - \bar{\partial}\gamma$, we see that $\bar{\partial}(\beta - \xi) = 0$. Hence, by the induction hypothesis, we may choose a smooth $(0, q-1)$ -form $\rho \in \Omega^{0,q-1}(\Delta)$ such that $(\beta - \xi)|_{\Delta} = \bar{\partial}\rho$. Setting $\eta := \rho + \gamma$, we see that

$$\bar{\partial}\eta = \bar{\partial}\rho + \bar{\partial}\gamma = (\beta - \xi)|_{\Delta} + (\omega - \beta + \xi)|_{\Delta} = \omega|_{\Delta},$$

as required. \square

Remark 3.2.11. Let $\mathcal{Z}_G^{0,1}$ denote the sheaf of $\bar{\partial}$ -closed $(0,1)$ -forms on a domain $G \subseteq \mathbb{C}^n$, and as usual, denote the sheaf of smooth functions on G by \mathcal{C}_G^∞ . By *Theorem 3.2.10*, the morphism $\bar{\partial} : \mathcal{C}_G^\infty \rightarrow \mathcal{Z}_G^{0,1}$ is stalkwise surjective. But if $\mathcal{D}^{0,1}(G)$ is non-zero, it is not true that the induced map $\bar{\partial} : \mathcal{C}_G^\infty(G) \rightarrow \mathcal{Z}_G^{0,1}(G)$ on sections is surjective. This provides an illustrative example of a surjection of sheaves which does not induce a surjection at the level of sections.

Definition 3.2.12. A *polydomain* in \mathbb{C}^n is understood as a domain of the form

$$G = G_1 \times \cdots \times G_n,$$

where G_k is a domain in \mathbb{C} for each $1 \leq k \leq n$.

Theorem 3.2.13. Let G be a simply-connected polydomain in \mathbb{C}^n . Then for all $p \in \mathbb{N}_0, q \in \mathbb{N}$, $\mathcal{D}^{p,q}(G) = 0$.

PROOF. By the Riemann mapping theorem of one complex variable, we may assume that G is the unit polydisk Δ centred at the origin in \mathbb{C}^n . We need to show that for every smooth (p,q) -form $\omega \in \Omega^{p,q}(\Delta)$, there exists a smooth $(p,q-1)$ -form $\eta \in \Omega^{p,q-1}(\Delta)$ such that $\omega = \bar{\partial}\eta$. To this end, we exhaust Δ by concentric polydisks $(\Delta_\nu)_{\nu \geq 1}$ in \mathbb{C}^n , i.e., $\bar{\Delta}_\nu \subset \Delta_{\nu+1}$ for each $\nu \geq 1$, and $\Delta = \bigcup_{\nu \geq 1} \Delta_\nu$, then prove the result in two separate cases: when $q > 1$, and when $q = 1$.

Suppose first that $q > 1$. For each $\nu \geq 1$, let U_ν be some open neighbourhood of $\bar{\Delta}_\nu$. We proceed inductively to construct a sequence $\eta_\nu \in \Omega^{p,q-1}(U_\nu)$ such that

- (i) $\omega = \bar{\partial}\eta_\nu$ on Δ_ν .
- (ii) $\eta_{\nu+1}|_{\Delta_\nu} = \eta_\nu$.

For $\nu = 1$ we need only apply *Theorem 3.2.10*, so assume that $\nu > 1$, and that we have chosen η_1, \dots, η_ν such that (i) and (ii) are satisfied. By *Theorem 3.2.10* there exists a smooth $(p,q-1)$ -form $\xi_{\nu+1} \in \Omega^{p,q-1}(U_{\nu+1})$ such that $\omega|_{U_{\nu+1}} = \bar{\partial}\xi_{\nu+1}$. On any open neighbourhood U_ν of $\bar{\Delta}_\nu$, the $(p,q-1)$ -form $\xi_{\nu+1} - \eta_\nu$ is $\bar{\partial}$ -closed since $\bar{\partial}(\xi_{\nu+1} - \eta_\nu) = \omega - \omega = 0$. Given that $q-1 > 0$, applying *Theorem 3.2.10*, there exists a smooth $(p,q-2)$ -form $\vartheta \in \Omega^{p,q-2}(U_\nu)$ such that $\xi_{\nu+1} - \eta_\nu = \bar{\partial}\vartheta$. Choose a smooth bump function $\rho \in \mathcal{C}^\infty(\mathbb{C}^n, \mathbb{R})$ such that

$$\rho(z) := \begin{cases} 1, & \forall z \in \bar{\Delta}_\nu, \\ 0, & \forall z \in \mathbb{C}^n \setminus \bar{U}_\nu. \end{cases}$$

Setting $\eta_{\nu+1} := \xi_{\nu+1} + \bar{\partial}(\rho \cdot \vartheta)$, we obtain a smooth $(p,q-1)$ -form on $U_{\nu+1}$ such that $\bar{\partial}\eta_{\nu+1} = \bar{\partial}\xi_{\nu+1} = \omega$ on $U_{\nu+1}$, and $\eta_{\nu+1}|_{\Delta_\nu} = \xi_{\nu+1}|_{\Delta_\nu} + \bar{\partial}\vartheta = \eta_\nu$, as required.

Now suppose that $q = 1$. We again proceed by induction on $\nu \geq 1$, but this time, we construct a sequence $\eta_\nu \in \Omega^{p,q-1}(U_\nu)$ such that

- (I) $\omega = \bar{\partial}\eta_\nu$ on Δ_ν .
- (II) the forms $\eta_{\nu+1} - \eta_\nu$ are holomorphic forms of bidegree $(p,0)$ on U_ν , which satisfy the estimate

$$|\eta_{\nu+1, j_1 \dots j_p}(z) - \eta_{\nu, j_1 \dots j_p}| < 2^{-\nu},$$

for all $z \in \bar{\Delta}_\nu$, and all coefficients (j_1, \dots, j_p) .

For $\nu = 1$, this is again immediate from *Theorem 3.2.10*. So suppose $\nu > 1$, and that η_1, \dots, η_ν have been constructed such that (I) and (II) are satisfied. By *Theorem 3.2.10* there exists a smooth $(p, q - 1)$ -form $\xi_{\nu+1} \in \Omega^{p, q-1}(U_{\nu+1})$ such that $\omega = \bar{\partial}\xi_{\nu+1}$. Moreover, since $\bar{\partial}(\xi_{\nu+1} - \eta_\nu) = 0$, the coefficients of $\xi_{\nu+1} - \eta_\nu$ are holomorphic on U_ν , and each coefficient admits a power series expansion centred at the origin which converges uniformly on $\bar{\Delta}_\nu$. Therefore, by appropriately choosing the partial sums, we obtain polynomials $f_{j_1 \dots j_p}(z)$ such that

$$|\xi_{\nu+1, j_1 \dots j_p}(z) - \eta_{\nu, j_1 \dots j_p}(z) - f_{j_1 \dots j_p}(z)| < 2^{-\nu},$$

for all $z \in \bar{\Delta}_\nu$. Set

$$f(z) := \sum_{j_1, \dots, j_p} f_{j_1 \dots j_p}(z) dz_{j_1} \wedge \dots \wedge dz_{j_p},$$

and define $\eta_{\nu+1} := \xi_{\nu+1} - f$. We observe that $\eta_{\nu+1}$ is a smooth $(p, q - 1)$ -form on $U_{\nu+1}$ with $\bar{\partial}\eta_{\nu+1} = \bar{\partial}\xi_{\nu+1} - \bar{\partial}f = \bar{\partial}\xi_{\nu+1} = \omega$ on $U_{\nu+1}$. Since $\bar{\partial}(\eta_{\nu+1} - \eta_\nu) = 0$, the smooth $(p, 0)$ -form $\eta_{\nu+1} - \eta_\nu$ is holomorphic on U_ν , and satisfies the desired estimate by construction.

Finally, for each $\nu \geq 1$, the coefficients of η_ν converge uniformly on any one of the polydisks Δ_ν to the coefficients of a smooth $(p, q - 1)$ -form η . Since $\eta - \eta_\mu = \lim_{\nu \rightarrow \infty} (\eta_\nu - \eta_\mu)$, and the forms $\eta_\nu - \eta_\mu$ are holomorphic on Δ_μ , it follows that $\eta = \eta_\mu + \vartheta_\mu$ for some holomorphic form ϑ_μ . Therefore, $\bar{\partial}\eta = \bar{\partial}\eta_\mu = \omega$ in each Δ_μ , and this completes the proof. \square

§3.3. CARTAN'S THEOREM B FOR STEIN SPACES

We have developed enough machinery to state Cartan's theorem *B* for Stein spaces. There is more theory to be developed, but we can make a good amount of progress on the proof with what we have already. Throughout this section, we assume that Cartan's theorem B holds for simply-connected polydomains in \mathbb{C}^n . A proof of this is given in §3.4. Moreover, we often abbreviate (X, \mathcal{O}_X) to X when X is understood to be a complex analytic space. Let us offer a statement of the main theorem.

Cartan's Theorem B. Let X be a Stein space, then for any coherent analytic sheaf \mathcal{S} on X , $H^q(X, \mathcal{S}) = 0$ for all $q \geq 1$.

The proof of Cartan's theorem *B* uses an argument of exhaustion type. That is, we will exhaust X by a suitable class of sets $(K_\nu)_{\nu \geq 1}$ on which theorem B holds, and bootstrap our way up to obtain the result on X , c.f., *Triviality 2.4.23*. The obvious first choice of exhaustion sets $(K_\nu)_{\nu \geq 1}$ is given in the following definition:

Definition 3.3.1. Let X be a complex analytic space, and K a closed subset of X . We say that K is a \mathcal{B} -space if Cartan's theorem B holds on K , i.e., for any coherent analytic sheaf \mathcal{S} on K , $H^q(K, \mathcal{S}) = 0$ for all $q \geq 1$.

Convention 3.3.2. Note that by \mathcal{S} being defined on a closed set K we mean that \mathcal{S} is the restriction to K of a sheaf defined on an open neighbourhood of K .

Let $(K_\nu)_{\nu \geq 1}$ be a sequence of \mathcal{B} -spaces which exhaust a complex analytic space X , i.e.,

$$X = \bigcup_{\nu \geq 1} K_\nu, \quad \text{and} \quad K_\nu \subset \overset{\circ}{K}_{\nu+1}, \quad \forall \nu \in \mathbb{N}.$$

The sheaf cohomology groups $H^q(X, \mathcal{S})$ of X are computed via a resolution of soft sheaves:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}^0 \longrightarrow \mathcal{S}^1 \longrightarrow \dots \longrightarrow \mathcal{S}^{q-1} \longrightarrow \mathcal{S}^q \longrightarrow \dots$$

The restriction of \mathcal{S} to K_ν yields the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{S}(X) & \xrightarrow{i} & \mathcal{S}^0(X) & \xrightarrow{d_0} & \mathcal{S}^1(X) & \xrightarrow{d_1} & \dots & \longrightarrow & \mathcal{S}^{q-1}(X) & \xrightarrow{d_{q-1}} & \mathcal{S}^q(X) & \longrightarrow & \dots \\ & & \downarrow \text{res}_\nu & & \downarrow \text{res}_\nu & & \downarrow \text{res}_\nu & & & & \downarrow \text{res}_\nu & & \downarrow \text{res}_\nu & & \\ 0 & \longrightarrow & \mathcal{S}(K_\nu) & \xrightarrow{i} & \mathcal{S}^0(K_\nu) & \xrightarrow{d_0^\nu} & \mathcal{S}^1(K_\nu) & \xrightarrow{d_1^\nu} & \dots & \longrightarrow & \mathcal{S}^{q-1}(K_\nu) & \xrightarrow{d_{q-1}^\nu} & \mathcal{S}^q(K_\nu) & \longrightarrow & \dots \end{array}$$

Observe that $H^q(K_\nu, \mathcal{S}) = \ker(d_q^\nu) / \text{im}(d_{q-1}^\nu) = 0$ if and only if the bottom row of the above diagram is exact at $\mathcal{S}^q(K_\nu)$, i.e., $\ker(d_q^\nu) = \text{im}(d_{q-1}^\nu)$. Moreover, for each $q \geq 0$, \mathcal{S}^q is a soft sheaf, and therefore any section in $\mathcal{S}^q(K_\nu)$ is the restriction to K_ν of a global section in $\mathcal{S}^q(X)$.

Proposition 3.3.3. Let X be a complex analytic space which admits an exhaustion by \mathcal{B} -spaces. Then for any coherent analytic sheaf \mathcal{S} on X , $H^q(X, \mathcal{S}) = 0$ for all $q \geq 2$.

PROOF. Let $(K_\nu)_{\nu \geq 1}$ be an exhaustion of X by \mathcal{B} -spaces. To show that $H^q(X, \mathcal{S}) = 0$ for all $q \geq 2$, we need to show that for any $\alpha \in \ker d_q$ there exists some $\beta \in \mathcal{S}^{q-1}(X)$ such that $d_{q-1}(\beta) = \alpha$. To achieve this, we construct a sequence $\beta_\nu \in \mathcal{S}^{q-1}(K_\nu)$ such that for each $\nu \geq 1$,

- (i) $d_{q-1}^\nu(\beta_\nu) = \text{res}_\nu(\alpha)$.
- (ii) $\text{res}_\nu(\beta_{\nu+1}) = \beta_\nu$.

We may then glue together these sections to obtain a global section $\beta \in \mathcal{S}^{q-1}(X)$ such that

$$\text{res}_\nu(\beta) = \beta_\nu \quad \text{and} \quad d_{q-1}^\nu(\beta_\nu) = \text{res}_\nu(\alpha).$$

Since $H^q(K_\nu, \mathcal{S}) = 0$ for each $\nu \geq 1$, we have a sequence $\beta'_\nu \in \mathcal{S}^{q-1}(K_\nu)$ with $d_{q-1}^\nu(\beta'_\nu) = \text{res}_\nu(\alpha)$. We proceed inductively to construct the desired sequence $(\beta_\nu)_{\nu \geq 1}$. Set $\beta_1 := \beta'_1$, and assume that $\beta_1, \dots, \beta_\nu$ have been chosen such that (i) and (ii) are satisfied. We compute

$$\begin{aligned} d_{q-1}^\nu(\text{res}_\nu(\beta'_{\nu+1}) - \beta_\nu) &= d_{q-1}^\nu(\text{res}_\nu(\beta'_{\nu+1})) - d_{q-1}^\nu(\beta_\nu) \\ &= \text{res}_\nu(d_{q-1}^{\nu+1}(\beta'_{\nu+1})) - \text{res}_\nu(\alpha) \\ &= \text{res}_\nu(\alpha) - \text{res}_\nu(\alpha) = 0, \end{aligned} \tag{21}$$

i.e., $\text{res}_\nu(\beta'_{\nu+1}) - \beta_\nu$ lies in the kernel of d_{q-1}^ν . Note that (21) follows from the commutativity of the above diagram. By the exactness of the bottom row of the diagram we choose $\gamma'_\nu \in \mathcal{S}^{q-2}(K_\nu)$ such that $d_{q-2}^\nu(\gamma'_\nu) = \text{res}_\nu(\beta'_{\nu+1}) - \beta_\nu$. For each $q \geq 2$, the sheaves \mathcal{S}^{q-2} are soft, so $\gamma'_\nu \in \mathcal{S}^{q-2}(K_\nu)$ is the restriction to K_ν of a global section $\gamma_\nu \in \mathcal{S}^{q-2}(X)$. Set $\beta_{\nu+1} := \beta'_{\nu+1} - \text{res}_{\nu+1}(d_{q-2}(\gamma_\nu))$; we claim that $\beta_{\nu+1}$ satisfies (i) and (ii).

Indeed, (i) follows from:

$$\begin{aligned}
d_{q-1}^{\nu+1}(\beta_{\nu+1}) &= d_{q-1}^{\nu+1}(\beta'_{\nu+1} - \text{res}_{\nu+1}(d_{q-2}(\gamma_\nu))) \\
&= d_{q-1}^{\nu+1}(\beta'_{\nu+1}) - d_{q-1}^{\nu+1}(\text{res}_{\nu+1}(d_{q-2}(\gamma_\nu))) \\
&= \text{res}_{\nu+1}(\alpha) - d_{q-1}^{\nu+1} \circ d_{q-2}^{\nu+1} \circ \text{res}_{\nu+1}(\gamma_\nu) \\
&= \text{res}_{\nu+1}(\alpha),
\end{aligned} \tag{22}$$

where (22) follows from the commutativity of the above diagram, and the last line follows from the fact that $d_{q-1}^{\nu+1} \circ d_{q-2}^{\nu+1} = 0$. To see that $\beta_{\nu+1}$ satisfies (ii), we simply observe that

$$\begin{aligned}
\text{res}_\nu(\beta_{\nu+1}) &= \text{res}_\nu(\beta'_{\nu+1} - d_{q-2}(\gamma_\nu)) \\
&= \text{res}_\nu(\beta'_{\nu+1}) - \text{res}_\nu(d_{q-2}(\gamma_\nu)) \\
&= \text{res}_\nu(\beta'_{\nu+1}) - (\text{res}_\nu(\beta'_{\nu+1}) - \beta_\nu) \\
&= \beta_\nu,
\end{aligned}$$

as required. □

Remark 3.3.4. In the proof of *Proposition 3.3.3*, we used the fact that for each $q \geq 2$, the sheaves \mathcal{S}^{q-2} were soft. This allowed us to extend any section in $\mathcal{S}^{q-2}(K_\nu)$ to a global section in $\mathcal{S}^{q-2}(X)$. This does not work for $H^1(X, \mathcal{S})$ however, since \mathcal{S} is not necessarily a soft sheaf. Therefore, to establish the vanishing of $H^1(X, \mathcal{S})$, a different approach must be taken. Indeed, in place of extending sections of $\mathcal{S}^{q-2}(K_\nu)$ to sections of $\mathcal{S}^{q-2}(X)$, we will endow $\mathcal{S}^{q-2}(X)$ with a suitable (Fréchet) topology such that for each $\nu \geq 1$, $\mathcal{S}(X)|_{K_\nu}$ is dense in $\mathcal{S}(K_\nu)$. This program will also require a refinement in the choice of exhaustion sets. The following class of exhaustion sets will be of more use.

Definition 3.3.5. Let X be a complex analytic space. An *analytic stone* in X is a quadruple (S, π, Q, W) consisting of a compact set $S \subset X$, a holomorphic map $\pi : X \rightarrow \mathbb{C}^m$, a compact (Euclidean) block² $Q \subset \mathbb{C}^m$, and an open set $W \subset X$ such that $\pi^{-1}(Q) = S \cap W$. We define the *analytic interior* of S to be the set $\mathring{A}(S) := \pi^{-1}(\mathring{Q}) \cap W$, where \mathring{Q} denotes the topological interior of Q .

Notational Remark 3.3.6. We will often abbreviate (S, π, Q, W) to (S, π) .

We consider the following elementary lemma from point-set topology.

Lemma 3.3.7. Let X and Y be two locally compact topological spaces, and $f : X \rightarrow Y$ a continuous map. If W an open relatively compact subset of X , then the induced map $g : W \setminus f^{-1}(f(\partial W)) \rightarrow Y \setminus f(\partial W)$ is a proper map.

PROOF. Let $K \subset Y \setminus f(\partial W)$ be a compact set, and let $\mathfrak{U} := \{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover for $f^{-1}(K)$. The set $V := \overline{W} \setminus f^{-1}(K)$ is open in \overline{W} , so $\mathfrak{U} \cup V$ is an open cover of \overline{W} . Since \overline{W} is compact, $\mathfrak{U} \cup V$ admits a finite subcover $\{U_{\lambda_1}, \dots, U_{\lambda_n}, V\}$. But this implies that $\{U_{\lambda_1}, \dots, U_{\lambda_n}\}$ is an open cover of $f^{-1}(K)$, which is a finite subcover of \mathfrak{U} , as required. □

²In other words, a product of rectangles $Q = Q_1 \times \dots \times Q_m$, where $Q_k := \{z_k = x_k + iy_k : a_k \leq x_k \leq b_k, c_k \leq y_k \leq d_k\}$. If we set $g_k := \frac{1}{2}(b_k - a_k)$, and $h_k := \frac{1}{2}(d_k - c_k)$, then $(g_1, h_1, g_2, h_2, \dots, g_m, h_m) \in \mathbb{C}^m$ is called the *centre of the block* Q .

Lemma 3.3.8. Let (S, π) be an analytic stone in a complex analytic space X . There exist open neighbourhoods U and V of S and Q respectively, such that $\pi(U) \subset V$, $S = \pi^{-1}(Q) \cap U$, and $\pi|_U : U \rightarrow V$ is a proper map.

PROOF. Let $W \subset X$ be the open set associated to (S, π) . Since S is compact, W may be taken to be relatively compact, and therefore ∂W and $\pi(\partial W)$ are compact sets. Moreover, since $\partial W \cap \pi^{-1}(Q) = \emptyset$, the open set $V := \mathbb{C}^m \setminus \pi(\partial W)$ yields an open neighbourhood of Q . The set $U := \pi^{-1}(V) \cap W = W \setminus \pi^{-1}(\pi(\partial W))$ is an open set in X with $\pi(U) \subset V$, and the restriction of π to U yields a proper map by *Lemma 3.3.7*. Further,

$$\pi^{-1}(Q) \cap U = \pi^{-1}(Q) \cap W \setminus (\pi^{-1}(Q) \cap \pi^{-1}(\pi(\partial W))).$$

Since $Q \cap \pi(\partial W) = \emptyset$, we see that $\pi^{-1}(Q) \cap \pi^{-1}(\pi(\partial W)) = \emptyset$, and therefore, $\pi^{-1}(Q) \cap U = S$. In particular, U is an open neighbourhood of S , and this completes the proof. \square

We want to exhaust a complex analytic space by these analytic stones. We thus need to make sense of an *inclusion of analytic stones*.

Definition 3.3.9. Let $(S_1, \pi_1) := (S_1, \pi_1, Q_1, W_1)$ and $(S_2, \pi_2) := (S_2, \pi_2, Q_2, W_2)$ be analytic stones in a complex analytic space X . We say that (S_1, π_1) is contained in (S_2, π_2) , and write $(S_1, \pi_1) \preceq (S_2, \pi_2)$, if

- (i) S_1 is contained in the analytic interior of S_2 , i.e., $S_1 \subset \mathring{A}(S_2)$.
- (ii) there exists an $n \in \mathbb{N}$ and $q \in \mathbb{C}^n$ such that $\mathbb{C}^{m_2} = \mathbb{C}^{m_1} \times \mathbb{C}^n$, and $Q_1 \times \{q\} \subset \mathring{Q}_2$.
- (iii) there exists a holomorphic map $\varphi : X \rightarrow \mathbb{C}^n$ such that $\pi_2(x) = (\pi_1(x), \varphi(x))$ for all $x \in X$.

Definition 3.3.10. A sequence of analytic stones $(S_\nu, \pi_\nu)_{\nu \geq 1}$ in a complex analytic space X is said to be an *exhaustion of X by analytic stones* if

- (i) for each $\nu \geq 1$, there is an inclusion $(S_\nu, \pi_\nu) \preceq (S_{\nu+1}, \pi_{\nu+1})$.
- (ii) $X = \bigcup_{\nu \geq 1} \mathring{A}(S_\nu)$.

Proposition 3.3.11. Let X be a complex analytic space, and let $K \subset X$ be a compact set. If there exists a relatively compact neighbourhood $W \subset X$ of K such that $\partial W \cap \widehat{K} = \emptyset$, then there exists an analytic stone (S, π) such that $K \subset \mathring{A}(S)$.

PROOF. Suppose that $\partial W \cap \widehat{K} = \emptyset$. By *Lemma 2.1.8*, for any point $p \in \partial W$ there exists an analytic function $f \in \mathcal{O}_X(X)$ such that $\|f\|_K < 1 < |f(p)|$. Since W is relatively compact, the boundary ∂W is compact, and we can choose a finite number of global sections $f_1, \dots, f_m \in \mathcal{O}_X(X)$ such that

$$\max_{1 \leq k \leq m} \left\{ \max_{z \in K} |\operatorname{Re}(f_k)|, \max_{z \in K} |\operatorname{Im}(f_k)| \right\} < 1. \quad (23)$$

and by raising the f_k to a suitable power if necessary,

$$\max_{1 \leq k \leq m} \{ |\operatorname{Re}(f_k)(p)|, |\operatorname{Im}(f_k)(p)| \} > 1, \quad \forall p \in \partial W. \quad (24)$$

Set $Q := \{(z_1, \dots, z_m) \in \mathbb{C}^m : |\operatorname{Re}(z_j)| \leq 1, |\operatorname{Im}(z_j)| \leq 1\}$ to be the standard unit block in \mathbb{C}^m and define a holomorphic map $\pi : X \rightarrow \mathbb{C}^m$ by setting $\pi(x) = (\operatorname{red}(f_1)(x), \dots, \operatorname{red}(f_m)(x))$. Then from (23) and (24), $\pi(\partial W) \cap Q = \emptyset$, and $K \subset \pi^{-1}(\mathring{Q}) \cap W$. In particular, we obtain an analytic stone (S, π) by setting $S := \pi^{-1}(Q) \cap W$, which is compact and whose analytic interior contains K , i.e., $K \subset \mathring{A}(S)$. \square

Theorem 3.3.12. Let X be a Stein space, then X admits an exhaustion by analytic stones.

PROOF. By *Proposition 2.4.22*, every complex analytic space X admits an exhaustion by compact sets $(K_\nu)_{\nu \geq 1}$. We proceed inductively to construct an exhaustion by analytic stones $(S_\nu, \pi_\nu)_{\nu \geq 1}$ such that $K_{\nu-1} \subset \mathring{A}(S_{\nu-1})$ for each $\nu \geq 2$. To this end, choose (S_1, π_1) in the obvious way, and suppose that $(S_{\nu-1}, \pi_{\nu-1})$ is an analytic stone with $K_{\nu-1} \subset \mathring{A}(S_{\nu-1})$. Since X is holomorphically convex, the holomorphically convex hull of $K_\nu \cup S_{\nu-1}$ is compact. By *Proposition 3.3.11* we construct an analytic stone (S_ν, π_ν^*) such that $K_\nu \cup S_{\nu-1} \subset \mathring{A}(S_\nu)$. Let $\pi_\nu^* : X \rightarrow \mathbb{C}^n$ be the holomorphic map, Q_ν^* to be the associated compact (Euclidean) block, and $W \subset X$ the open set such that $S_\nu = (\pi_\nu^*)^{-1}(Q_\nu^*) \cap W$. Choose a compact (Euclidean) block $Q'_\nu \subset \mathbb{C}^{m_{\nu-1}}$ such that $Q_{\nu-1}$ and $\pi_{\nu-1}(S_\nu)$ are both contained in \mathring{Q}'_ν . Set $\pi_\nu = (\pi_{\nu-1}, \pi_\nu^*) : X \rightarrow \mathbb{C}^{m_{\nu-1}} \times \mathbb{C}^n$, where $Q_\nu = Q'_\nu \times Q_\nu^*$. We claim that (S_ν, π_ν) is the desired analytic stone. Indeed,

$$\pi_\nu^{-1}(Q_\nu) \cap W = \pi_{\nu-1}^{-1}(Q'_\nu) \cap ((\pi_\nu^*)^{-1}(Q_\nu^*) \cap W) = \pi_{\nu-1}^{-1}(Q'_\nu) \cap S_\nu = S_\nu,$$

where the last equality follows from the fact that $\pi_{\nu-1}(S_\nu) \subset \mathring{Q}'_\nu$. In particular $\pi_{\nu-1}^{-1}(Q'_\nu)$ is contained in the analytic interior $\mathring{A}(S_\nu)$. Finally, given that $X = \bigcup_{\nu \geq 1} K_\nu \subset \bigcup_{\nu \geq 1} \mathring{A}(S_\nu)$, it follows that X may be exhausted by analytic stones. \square

Definition 3.3.13. Let (S, π) be an analytic stone in a complex analytic space. By *Lemma 3.3.8*, there exist open neighbourhoods U and V of S and W respectively, such that $\pi|_U : U \rightarrow V$ is a proper map. If U and V can be chosen such that $\pi|_U : U \rightarrow V$ is a finite map, then (S, π) is said to be an *analytic block*.

Proposition 3.3.14. Let (S, π) be an analytic block in a complex analytic space X , then S is a \mathcal{B} -space.

PROOF. Let $\tau := \pi|_U : U \rightarrow V$ be the finite map specified by the definition of an analytic block, and let \mathcal{S} be a coherent analytic sheaf on S . Choose the neighbourhood $U \supset S$ sufficiently small such that \mathcal{S} is coherent on U . The restriction of a finite map remains finite and we identify V with a smaller open neighbourhood induced from this restriction of U . By *Theorem 3.1.32* the direct image sheaf $\mathcal{T} := \tau_* \mathcal{S}$ is a coherent analytic sheaf on V , and by *Theorem 3.1.31* there is an induced isomorphism of cohomology groups $H^q(S, \mathcal{S}) \cong H^q(Q, \mathcal{T})$. By Cartan's theorem *B* for simply-connected polydomains (see *Theorem 3.4.10*) $H^q(S, \mathcal{S}) \cong H^q(Q, \mathcal{T}) = 0$ for all $q \geq 1$, and so S is a \mathcal{B} -space. \square

Definition 3.3.15. An exhaustion of a complex analytic space X by analytic stones $(S_\nu, \pi_\nu)_{\nu \geq 1}$ is said to be an *exhaustion by analytic blocks* if each analytic stone (S_ν, π_ν) is an analytic block.

Proposition 3.3.16. Let X be a Stein space, then X admits an exhaustion by analytic blocks $(S_\nu, \pi_\nu)_{\nu \geq 1}$.

PROOF. Let X be a Stein space. By *Theorem 3.3.12* X admits an exhaustion by analytic stones. For any analytic stone (S_ν, π_ν) in this exhaustion let U_ν and V_ν be the respective open neighbourhoods of S_ν and Q_ν such that $\tau_\nu := \pi|_{U_\nu} : U_\nu \rightarrow V_\nu$ is a proper map. We want to show that τ_ν is finite for each $\nu \geq 1$. To this end, choose a point $y \in V_\nu$ and let H_ν be a connected component of the fibre $\tau_\nu^{-1}(y) \subset X$. Since τ_ν is proper $\tau_\nu^{-1}(y)$ is compact, and by the maximum principle, the restriction to H_ν of any $f \in \mathcal{O}_X(X)$ is constant. Since X is Stein however, X is holomorphically separable and this can only occur if H consists of a single point. Finally, since a compact set has only a finite number of connected components it follows that τ_ν is finite. \square

Remark 3.3.17. By this point, the reader may be overwhelmed by the technical nature of the arguments. We wish to offer the reader with some intuition to lift their head above the trees and regain sight of the forrest. Our goal is to prove that every Stein space (X, \mathcal{O}_X) is a \mathcal{B} -space. By *Proposition 3.3.3*, if we exhaust X by \mathcal{B} -spaces, $H^q(X, \mathcal{S}) = 0$ for all $q \geq 2$. The problem is therefore concentrated in showing that $H^1(X, \mathcal{S}) = 0$. The prototypical \mathcal{B} -space, for us at least, is a compact block³ $Q \subset \mathbb{C}^m$. Our plan of attack is therefore to construct an exhaustion $(K_\nu)_{\nu \geq 1}$ of X which hijacks the results of the prototype case, and establishes them on X . The desired exhaustion should satisfy the following:

- (i) Each K_ν is a \mathcal{B} -space.
- (ii) We may endow each $\mathcal{S}(K_\nu)$ with a suitable Fréchet topology to approximate sections in $\mathcal{S}(X)|_{K_\nu}$ by sections in $\mathcal{S}(K_\nu)$, c.f., *Remark 3.3.4*.
- (iii) A Stein space must admit an exhaustion by these sets.

By *Theorem 3.3.12*, any holomorphically convex complex analytic space admits an exhaustion by analytic stones. But *Lemma 3.3.8* only ensures that $\pi|_U : U \rightarrow V$ is a proper map, so *Theorem 3.1.31* is not readily available. To circumvent this problem, we simply define analytic blocks to be exactly those analytic stones which ensure that (i) is satisfied. By *Proposition 3.3.16*, a Stein space admits an exhaustion by analytic blocks, so an exhaustion by analytic blocks satisfies (iii) also. The fact that an exhaustion by analytic blocks satisfies (ii) is the content of §4.1 (see *Theorem 4.1.5*).

Let us note that in the proof of *Proposition 3.3.16* we have used both the assumption of holomorphic convexity and holomorphic separability. Therefore, despite the fact that a compact complex manifold will admit an exhaustion by analytic stones, it will not admit an exhaustion by analytic blocks, c.f., *Lemma 2.1.8*, and *Remark 2.1.15*.

§3.4. CARTAN'S THEOREM B FOR SIMPLY-CONNECTED POLYDOMAINS IN \mathbb{C}^n .

Let $G \subseteq \mathbb{C}^n$ be a simply-connected polydomain, and let \mathcal{S} be a coherent analytic sheaf on G . The purpose of this section is to show that $H^q(G, \mathcal{S}) = 0$ for all $q \geq 1$. We follow the argument in Chapter 6 of [26], which first appeared in [25], and centres around the notion of a syzygy which was discussed in §2.3.

We remind the reader of the definition. Indeed, a chain of syzygies of length k for an analytic sheaf \mathcal{S} is an exact sequence of the form

$$0 \longrightarrow \mathcal{O}^{p_k} \xrightarrow{\lambda_k} \mathcal{O}^{p_{k-1}} \longrightarrow \dots \longrightarrow \mathcal{O}^{p_1} \xrightarrow{\lambda_1} \mathcal{O}^p \xrightarrow{\lambda} \mathcal{S} \longrightarrow 0, \quad (25)$$

where $p_k \in \mathbb{N}$.

Lemma 3.4.1. Let G be a simply-connected polydomain in \mathbb{C}^n , then $H^q(G, \mathcal{O}) = 0$ for all $q \geq 1$.

PROOF. This is an immediate consequence of *Theorem 3.2.13* and *Theorem 3.2.8*. □

³The fact that Q is a \mathcal{B} -space is proved in the next section (see *Theorem 3.4.10*).

Proposition 3.4.2. Let $G \subseteq \mathbb{C}^n$ be a simply-connected polydomain and \mathcal{S} an analytic sheaf over G . If \mathcal{S} admits a terminating chain of syzygies over G , then $H^q(G, \mathcal{S}) = 0$ for all $q \geq 1$.

PROOF. Let \mathcal{S} be an analytic sheaf over G which admits a terminating chain of syzygies of the form (25). We will proceed by induction on the length k of the syzygy (25). For $k = 0$ (25) is just the exact sequence $0 \rightarrow \mathcal{O}^p \rightarrow \mathcal{S} \rightarrow 0$. In other words, there is an isomorphism $\mathcal{O}^p \cong \mathcal{S}$. If $p = 1$ then *Lemma 3.4.1* informs us that $H^q(G, \mathcal{S}) = 0$, therefore, assume that $p > 1$. By the exactness of the sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}^p \rightarrow \mathcal{O}^{p-1} \rightarrow 0$$

we have an induced exact sequence on cohomology

$$H^q(G, \mathcal{O}) \rightarrow H^q(G, \mathcal{O}^p) \rightarrow H^q(G, \mathcal{O}^{p-1}) \rightarrow H^{q+1}(G, \mathcal{O}).$$

Then by *Lemma 3.4.1* $H^q(G, \mathcal{O}) \cong H^{q+1}(G, \mathcal{O}) = 0$. Therefore $H^q(G, \mathcal{O}^p) \cong H^q(G, \mathcal{O}^{p-1})$ for all $p \geq 2$ and $q \geq 1$. Now assume the result holds for all syzygies of lengths at most $k - 1$. The syzygy (25) may be severed into two exact sequences

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}^p \xrightarrow{\lambda} \mathcal{S} \rightarrow 0 \quad (26)$$

and

$$0 \rightarrow \mathcal{O}^{p_k} \rightarrow \dots \rightarrow \mathcal{O}^p \rightarrow \mathcal{F} \rightarrow 0,$$

where $\mathcal{F} = \mathcal{K}er(\lambda)$. The induction hypothesis informs us that $H^q(G, \mathcal{F}) = 0$ for all $q \geq 1$. Therefore, from (26), we have an isomorphism between $H^q(G, \mathcal{S})$ and $H^q(G, \mathcal{O}^p)$ for all $q \geq 1$. We showed above however that $H^q(G, \mathcal{O}^p) = 0$ for all $p, q \geq 1$, so $H^q(G, \mathcal{S}) = 0$, and this completes the proof. \square

Corollary 3.4.3. Let $G \subseteq \mathbb{C}^n$ be a simply-connected polydomain. Associated to any terminating chain of syzygies over G of the form (25) there is an associated exact sequence

$$0 \rightarrow \mathcal{O}^{p_k}(G) \xrightarrow{\lambda_k} \mathcal{O}^{p_{k-1}}(G) \rightarrow \dots \rightarrow \mathcal{O}^{p_1}(G) \xrightarrow{\lambda_1} \mathcal{O}^p(G) \xrightarrow{\lambda} \mathcal{S}(G) \rightarrow 0, \quad (27)$$

at the level of sections.

PROOF. We proceed by induction on the length k of the syzygy (25). If $k = 0$ the claim is trivial, so assume that $k > 0$ and that the result holds for all syzygies of length at most $k - 1$. Just as we did in the proof of *Proposition 3.4.2*, write (25) as two shorter exact sequences

$$0 \rightarrow \mathcal{O}^{p_k} \rightarrow \dots \rightarrow \mathcal{O}^p \rightarrow \mathcal{F} \rightarrow 0, \quad (28)$$

and

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}^p \xrightarrow{\lambda} \mathcal{S} \rightarrow 0 \quad (29)$$

with $\mathcal{F} = \mathcal{K}er(\lambda)$. By the induction hypothesis (28) induces an exact sequence on sections

$$0 \rightarrow \mathcal{F}(G) \rightarrow \mathcal{O}^p(G) \xrightarrow{\lambda} \mathcal{S}(G) \rightarrow 0.$$

Further, by the exact sequence on cohomology associated to (29), the following sequence is also exact

$$0 \rightarrow \mathcal{F}(G) \rightarrow \mathcal{O}^p(G) \rightarrow \mathcal{S}(G) \rightarrow H^1(G, \mathcal{S}) \rightarrow \dots \quad (30)$$

Then by (3), $H^1(G, \mathcal{S}) = 0$, and by combining (29) and (30), it follows that (27) is exact. \square

Definition 3.4.4. Let $G \subseteq \mathbb{C}^m$ be a domain and \mathcal{S} an analytic sheaf over G . Let

$$\mathcal{O}^{p_m} \xrightarrow{\lambda_m} \mathcal{O}^{p_{m-1}} \longrightarrow \dots \longrightarrow \mathcal{O}^{p_1} \xrightarrow{\lambda_1} \mathcal{O}^p \xrightarrow{\lambda} \mathcal{S} \longrightarrow 0 \quad (31)$$

be a chain of syzygies for \mathcal{S} . A *modification* of (31) at the k th place, $1 \leq k \leq m$, is a syzygy for \mathcal{S} of the form

$$\begin{aligned} \mathcal{O}^{p_m} \xrightarrow{\lambda_m} \mathcal{O}^{p_{m-1}} \longrightarrow \dots \longrightarrow \mathcal{O}^{p_{k+1}} \xrightarrow{\lambda_{k+1}} \mathcal{O}^{p_{k+q}} \xrightarrow{\widetilde{\lambda}_k} \mathcal{O}^{p_{k-1+q}} \xrightarrow{\widetilde{\lambda}_{k+1}} \mathcal{O}^{p_{k-1}} \longrightarrow \dots \\ \dots \longrightarrow \mathcal{O}^{p_1} \xrightarrow{\lambda_1} \mathcal{O}^p \xrightarrow{\lambda} \mathcal{S} \longrightarrow 0, \end{aligned}$$

where the maps $\widetilde{\lambda}_k$ and $\widetilde{\lambda}_{k+1}$ are defined in the obvious way.

Definition 3.4.5. Let G be a domain in \mathbb{C}^n , and suppose that over G we have two coherent analytic sheaves \mathcal{S} and \mathcal{T} which admits chains of syzygies of the form

$$\mathcal{O}^{p_m} \xrightarrow{\lambda_m} \mathcal{O}^{p_{m-1}} \longrightarrow \dots \longrightarrow \mathcal{O}^{p_1} \xrightarrow{\lambda_1} \mathcal{O}^{p_0} \xrightarrow{\lambda} \mathcal{S} \longrightarrow 0, \quad (32)$$

$$\mathcal{O}^{q_m} \xrightarrow{\mu_m} \mathcal{O}^{q_{m-1}} \longrightarrow \dots \longrightarrow \mathcal{O}^{q_1} \xrightarrow{\mu_1} \mathcal{O}^{q_0} \xrightarrow{\mu} \mathcal{T} \longrightarrow 0. \quad (33)$$

Let $\varphi : \mathcal{S} \rightarrow \mathcal{T}$ be a morphism of analytic sheaves. A *morphism of syzygies lying over φ* is a collection of morphisms of analytic sheaves $\varphi_k : \mathcal{O}^{p_k} \rightarrow \mathcal{O}^{q_k}$ such that the diagram

$$\begin{array}{ccccccccccc} \mathcal{O}^{p_m} & \xrightarrow{\lambda_m} & \mathcal{O}^{p_{m-1}} & \longrightarrow & \dots & \longrightarrow & \mathcal{O}^{p_1} & \xrightarrow{\lambda_1} & \mathcal{O}^{p_0} & \xrightarrow{\lambda} & \mathcal{S} & \longrightarrow & 0 \\ \downarrow \varphi_m & & \downarrow \varphi_{m-1} & & & & \downarrow \varphi_1 & & \downarrow \varphi_0 & & \downarrow \varphi & & \\ \mathcal{O}^{q_m} & \xrightarrow{\mu_m} & \mathcal{O}^{q_{m-1}} & \longrightarrow & \dots & \longrightarrow & \mathcal{O}^{q_1} & \xrightarrow{\mu_1} & \mathcal{O}^{q_0} & \xrightarrow{\mu} & \mathcal{T} & \longrightarrow & 0 \end{array} \quad (34)$$

is commutative. If φ and all the φ_j are isomorphisms of analytic sheaves, we say that we have an *isomorphism of syzygies lying over φ* .

Proposition 3.4.6. Suppose that G is a simply-connected polydomain in \mathbb{C}^n , and let \mathcal{S} , \mathcal{T} be two analytic sheaves which admit terminating chains of syzygies over G . If $\varphi : \mathcal{S} \rightarrow \mathcal{T}$ is an isomorphism of sheaves, then after a finite number of modifications, there will exist an isomorphism between the two chains of syzygies lying over φ .

PROOF. Suppose that the analytic sheaves \mathcal{S} and \mathcal{T} admit terminating chains of syzygies of the form

$$0 \longrightarrow \mathcal{O}^{p_m} \xrightarrow{\mu_m} \dots \longrightarrow \mathcal{O}^{p_1} \xrightarrow{\mu_1} \mathcal{O}^p \xrightarrow{\mu} \mathcal{S} \longrightarrow 0 \quad (35)$$

and

$$0 \longrightarrow \mathcal{O}^{q_r} \xrightarrow{\nu_r} \dots \longrightarrow \mathcal{O}^{q_1} \xrightarrow{\nu_1} \mathcal{O}^q \xrightarrow{\nu} \mathcal{T} \longrightarrow 0. \quad (36)$$

We proceed by induction on $\ell := \max(m, r)$. Indeed, for $\ell = 0$, (35) and (36) reduce to

$$0 \longrightarrow \mathcal{O}^p \xrightarrow{\mu} \mathcal{S} \longrightarrow 0, \quad \text{and} \quad 0 \longrightarrow \mathcal{O}^q \xrightarrow{\nu} \mathcal{T} \longrightarrow 0,$$

i.e., $\mathcal{S} \cong \mathcal{O}^p$ and $\mathcal{T} \cong \mathcal{O}^q$. It is then clear that an isomorphism $\varphi : \mathcal{S} \rightarrow \mathcal{T}$ induces an isomorphism $\varphi_0 : \mathcal{O}^p \rightarrow \mathcal{O}^q$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}^p & \xrightarrow{\mu} & \mathcal{S} & \longrightarrow & 0 \\ & & \downarrow \varphi_0 & & \downarrow \varphi & & \\ 0 & \longrightarrow & \mathcal{O}^q & \xrightarrow{\nu} & \mathcal{T} & \longrightarrow & 0 \end{array}$$

is commutative. Now suppose the theorem holds for all pairs of syzygies of lengths at most $\ell - 1$. Let $e_1, \dots, e_p \in \mathcal{O}^p(G)$ denote the canonical generators of \mathcal{O}^p . The images $\varphi \circ \mu(e_k) \in \mathcal{T}(G)$ are well-defined sections, and by *Corollary 3.4.3*, since G is a simply-connected polydomain, we may choose $f_1, \dots, f_p \in \mathcal{O}^q(G)$ such that $\nu(f_k) = \varphi \circ \mu(e_k)$ for each $1 \leq k \leq p$. Hence, we may define a map $\sigma : \mathcal{O}^p \rightarrow \mathcal{O}^q$ such that $\sigma(e_k) = f_k$ for each k . Since e_1, \dots, e_p are the generators of \mathcal{O}^p , and $\nu \circ \sigma(e_k) = \nu(f_k) = \varphi \circ \mu(e_k)$ for each k , it follows that $\nu \circ \sigma = \varphi \circ \mu$. Applying this same argument to $\varphi^{-1} : \mathcal{T} \rightarrow \mathcal{S}$, we obtain a map $\tau : \mathcal{O}^q \rightarrow \mathcal{O}^p$ such that $\mu \circ \tau = \varphi^{-1} \circ \nu$. In pictures, the diagram

$$\begin{array}{ccccccc} \mathcal{O}^{p_1} & \xrightarrow{\mu_1} & \mathcal{O}^p & \xrightarrow{\mu} & \mathcal{S} & \longrightarrow & 0 \\ & & \tau \uparrow & \searrow \sigma & \downarrow \varphi & & \\ \mathcal{O}^{q_1} & \xrightarrow{\nu_1} & \mathcal{O}^q & \xrightarrow{\nu} & \mathcal{T} & \longrightarrow & 0 \end{array} \quad (37)$$

is commutative. If we now make the obvious modifications of (35) and (36) at the first stage, we obtain the diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathcal{O}^{p_2} & \xrightarrow{\mu'_2} & \mathcal{O}^{p_1} & \xrightarrow{\mu'_1} & \mathcal{O}^p & \xrightarrow{\mu'} & \mathcal{S} & \longrightarrow & 0 \\ & & & & \varphi'_0 \uparrow & \searrow \varphi_0 & \downarrow \varphi & & & & \\ \dots & \longrightarrow & \mathcal{O}^{q_2} & \xrightarrow{\nu'_2} & \mathcal{O}^{q_1} & \xrightarrow{\nu'_1} & \mathcal{O}^q & \xrightarrow{\nu'} & \mathcal{T} & \longrightarrow & 0, \end{array} \quad (38)$$

where $\varphi_0 : \mathcal{O}^p \oplus \mathcal{O}^q \rightarrow \mathcal{O}^q \oplus \mathcal{O}^p$ and $\varphi'_0 : \mathcal{O}^q \oplus \mathcal{O}^p \rightarrow \mathcal{O}^p \oplus \mathcal{O}^q$ are defined as follows. For each $x \in G$,

$$(\mathcal{O}_x^p \oplus \mathcal{O}_x^q) \ni \begin{pmatrix} f_x \\ g_x \end{pmatrix} \mapsto \varphi_0(f_x, g_x) = \begin{pmatrix} g_x - \sigma \circ \tau(g_x) + \sigma(f_x) \\ f_x - \tau(g_x) \end{pmatrix} \in (\mathcal{O}_x^q \oplus \mathcal{O}_x^p),$$

and

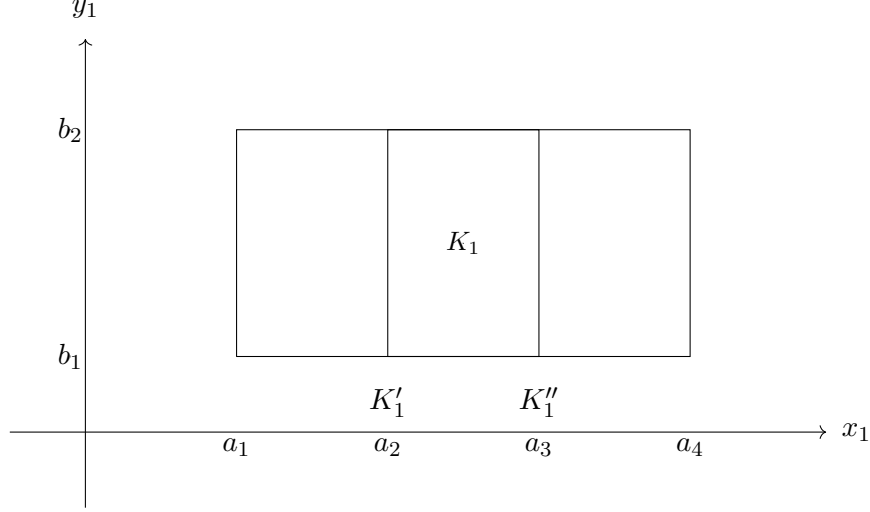
$$(\mathcal{O}_x^q \oplus \mathcal{O}_x^p) \ni \begin{pmatrix} s_x \\ t_x \end{pmatrix} \mapsto \varphi'_0(s_x, t_x) = \begin{pmatrix} t_x - \tau \circ \sigma(t_x) + \tau(s_x) \\ s_x - \sigma(t_x) \end{pmatrix} \in (\mathcal{O}_x^p \oplus \mathcal{O}_x^q).$$

Since (37) is commutative, (38) is commutative by construction. Further, it can be readily verified that $\varphi'_0 \circ \varphi_0 : \mathcal{O}^p \oplus \mathcal{O}^q \rightarrow \mathcal{O}^p \oplus \mathcal{O}^q$ is the identity map, and therefore φ_0 is an isomorphism. Set $\mathcal{S}_0 := \mathcal{H}er(\mu')$, $\mathcal{T}_0 := \mathcal{H}er(\nu')$, and observe that from (38), we have two exact sequences with a vertical isomorphism:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}^{p_m} & \xrightarrow{\mu_m} & \dots & \longrightarrow & \mathcal{O}^{p_2} & \xrightarrow{\mu'_2} & \mathcal{O}^{p_1+q} & \xrightarrow{\mu'_1} & \mathcal{S}_0 & \longrightarrow & 0 \\ & & & & & & & & & & \downarrow \varphi_0 & & \\ 0 & \longrightarrow & \mathcal{O}^{q_r} & \xrightarrow{\nu_r} & \dots & \longrightarrow & \mathcal{O}^{q_2} & \xrightarrow{\nu'_2} & \mathcal{O}^{q_1+p} & \xrightarrow{\nu'_1} & \mathcal{T}_0 & \longrightarrow & 0 \end{array} \quad (39)$$

By the induction hypothesis, after a finite number of modifications, we have an isomorphism between the two terminating chains of syzygies, lying over φ_0 . Combining the diagrams (38) and (39) completes the induction step, and this finishes the proof. \square

The following result due to H. Cartan [8] on holomorphic matrices will help us to glue syzygies together. To assist the reader we offer the following diagram for reference.



With the above diagram taken into consideration, let $a_1 < a_2 < a_3 < a_4$, $b_1 < b_2$ be real numbers. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ we have $z_k = x_k + iy_k$, where $x_k, y_k \in \mathbb{R}$ for all $1 \leq k \leq n$. Moreover, we set

$$\begin{aligned} K_1 &:= \{z_1 \in \mathbb{C} : a_2 < x_1 < a_3, b_1 < y_1 < b_2\}, \\ K_1' &:= \{z_1 \in \mathbb{C} : a_1 < x_1 < a_3, b_1 < y_1 < b_2\}, \\ K_1'' &:= \{z_1 \in \mathbb{C} : a_2 < x_1 < a_4, b_1 < y_1 < b_2\}. \end{aligned}$$

Note that $K_1' \cap K_1'' = K_1$. Then choose K_2, \dots, K_n to be simply-connected domains in \mathbb{C} with coordinates z_2, \dots, z_n , respectively. We now set

$$\begin{aligned} K &:= K_1 \times K_2 \times \cdots \times K_n, \\ K' &:= K_1' \times K_2 \times \cdots \times K_n, \\ K'' &:= K_1'' \times K_2 \times \cdots \times K_n. \end{aligned}$$

We may now state Cartan's lemma on holomorphic matrices. Given the length of the proof, we offer only the statement (see, e.g., [8], [26, p. 199] for a proof).

Theorem 3.4.7. (*Cartan*). If $F(z)$ is a holomorphic non-singular matrix in an open neighbourhood of K as defined above, then there exist holomorphic non-singular matrices F' and F'' on K' and K'' , respectively, such that $F(z) = F'(z) \cdot F''(z)$ for all $z \in K$.

Maintaining the above notation, we consider the following lemma:

Lemma 3.4.8. Let \mathcal{S} be an analytic sheaf on $\overline{K'} \cup \overline{K''}$ which admits a terminating chain of syzygies over $\overline{K'}$. Assume that \mathcal{S} also admits a terminating chain of syzygies over $\overline{K''}$. Then \mathcal{S} admits a terminating chain of syzygies over $\overline{K'} \cup \overline{K''}$.

PROOF. This is a straightforward consequence of *Proposition 3.4.6* and *Theorem 3.4.7*. Indeed, we first observe that by enlarging K , K' , and K'' , it suffices to show that \mathcal{S} admits a terminating chain of syzygies over the polydomain $K' \cup K''$. Let U' and U'' be respective open neighbourhoods of $\overline{K'}$ and $\overline{K''}$ such that

$U := U' \cap U''$ is also a polydomain. Suppose that \mathcal{S} admits a terminating chain of syzygies over U' and over U'' . Over U , this yields two terminating chains of syzygies of $\mathcal{S}|_U$. Then by *Proposition 3.4.6*, after a finite number of modifications, we have an isomorphism between these syzygies lying over the identity $\text{Id}_U : \mathcal{S}|_U \rightarrow \mathcal{S}|_U$. From *Definition 3.4.4*, we see that a modification over $U \subset U'$ can be extended to a modification over U' , and after performing these modifications, we arrive at the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{O}|_{U'})^{p_m} & \xrightarrow{\mu_m} & \cdots & \xrightarrow{\mu_1} & (\mathcal{O}|_{U'})^p & \xrightarrow{\mu} & (\mathcal{S}|_{U'}) & \longrightarrow & 0 \\ & & \downarrow \lambda_m & & & & \downarrow \lambda_0 & & \downarrow \text{Id}_U & & \\ 0 & \longrightarrow & (\mathcal{O}|_{U''})^{p_m} & \xrightarrow{\nu_m} & \cdots & \xrightarrow{\nu_1} & (\mathcal{O}|_{U''})^p & \xrightarrow{\nu} & (\mathcal{S}|_{U''}) & \longrightarrow & 0. \end{array} \quad (40)$$

The maps $\lambda_0, \dots, \lambda_m$ are isomorphisms on U which render (40) commutative when restricted to U (hence why the vertical arrows are dashed). Each λ_k is represented by a non-singular, holomorphic, matrix-valued function $F_k(z)$ on U . Hence, by *Theorem 3.4.7*, there are non-singular, holomorphic, matrix-valued functions $F'_k(z)$ and $F''_k(z)$ defined on K' and K'' respectively, such that

$$F_k(z) = [F''_k(z)]^{-1}[F'_k(z)]^{-1}, \quad \forall z \in K.$$

These matrices define isomorphisms $\lambda'_k : (\mathcal{O}|_{K'})^{p_k} \rightarrow (\mathcal{O}|_{K'})^{p_k}$ and $\lambda''_k : (\mathcal{O}|_{K''})^{p_k} \rightarrow (\mathcal{O}|_{K''})^{p_k}$, such that $\lambda_k = (\lambda''_k)^{-1}(\lambda'_k)^{-1}$ over K . In pictures, we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{O}|_{K'})^{p_m} & \xrightarrow{\mu'_m} & \cdots & \xrightarrow{\mu'_1} & (\mathcal{O}|_{K'})^p & \xrightarrow{\mu'} & (\mathcal{S}|_{K'}) & \longrightarrow & 0 \\ & & \downarrow \lambda'_m & & & & \downarrow \lambda'_0 & & \downarrow \text{Id} & & \\ 0 & \longrightarrow & (\mathcal{O}|_{K'})^{p_m} & \xrightarrow{\mu_m} & \cdots & \xrightarrow{\mu_1} & (\mathcal{O}|_{K'})^p & \xrightarrow{\mu} & (\mathcal{S}|_{K'}) & \longrightarrow & 0 \\ & & \downarrow \lambda_m & & & & \downarrow \lambda_0 & & \downarrow \text{Id} & & \\ 0 & \longrightarrow & (\mathcal{O}|_{K''})^{p_m} & \xrightarrow{\nu_m} & \cdots & \xrightarrow{\nu_1} & (\mathcal{O}|_{K''})^p & \xrightarrow{\nu} & (\mathcal{S}|_{K''}) & \longrightarrow & 0 \\ & & \downarrow \lambda''_m & & & & \downarrow \lambda''_0 & & \downarrow \text{Id} & & \\ 0 & \longrightarrow & (\mathcal{O}|_{K''})^{p_m} & \xrightarrow{\nu'_m} & \cdots & \xrightarrow{\nu'_1} & (\mathcal{O}|_{K''})^p & \xrightarrow{\nu'} & (\mathcal{S}|_{K''}) & \longrightarrow & 0, \end{array} \quad (41)$$

where, as before, the mappings $\lambda_0, \dots, \lambda_m$ are defined only on $U \supset K' \cup K''$. Moreover, the maps $\mu', \mu'_1, \dots, \mu'_m, \nu', \nu'_1, \dots, \nu'_m$ are the uniquely determined morphisms such that (41) is commutative when restricted to U . Upon restriction to $K = K' \cap K''$, rows 1 and 4 of (41) are isomorphic chains of syzygies lying over the identity map $\text{Id}_K : \mathcal{S}|_K \rightarrow \mathcal{S}|_K$ under the sheaf morphisms $\lambda''_k \lambda_k \lambda'_k : (\mathcal{O}|_K)^{p_k} \rightarrow (\mathcal{O}|_K)^{p_k}$. By construction however, $\lambda''_k \lambda_k \lambda'_k$ is simply the identity map over K . Therefore the syzygies coincide identically over K and we have a single terminating chain of syzygies over $K' \cup K''$, as required. \square

Theorem 3.4.9. (*Amalgamation of Syzygies Theorem*). Let $G \subseteq \mathbb{C}^n$ be a simply-connected polydomain in \mathbb{C}^n , and let $K \subset G$ be a compact set. There exists an open set $U \subseteq \mathbb{C}^n$ with $K \subset U \subset \bar{U} \subset G$, such that any coherent analytic sheaf \mathcal{S} over G has a terminating chain of syzygies over U .

PROOF. By the Riemann mapping theorem of one complex variable, we may assume that G is an open (Euclidean) block⁴ in \mathbb{C}^n . Choose U to be an open (Euclidean) block such that $K \subset U \subset \bar{U} \subset G$, and write $U = U_1 \times \cdots \times U_n$, where $U_k \subset \mathbb{C}$ is a rectangle with coordinate z_k . By *Theorem 2.3.6*, for any point

⁴That is, $G = G_1 \times \cdots \times G_n$, where $G_k := \{z_k \in \mathbb{C} : a_k < \text{Re}(z_k) < b_k, c_k < \text{Im}(z_k) < d_k\}$.

$p \in G$, there exists an open neighbourhood of p over which \mathcal{S} admits a terminating chain of syzygies. We therefore decompose the (Euclidean) block U into a finite number of (Euclidean) blocks such that \mathcal{S} admits a terminating chain of syzygies of each of these sub-blocks. Then by applying *Lemma 3.4.8* iteratively, we obtain a terminating chain of syzygies over all of U . The details of this construction are given in [26, p. 205–206] and omitted in the interests of brevity. \square

Theorem 3.4.10. (*Cartan*). Let $G \subseteq \mathbb{C}^n$ be a simply-connected polydomain and \mathcal{S} a coherent analytic sheaf over G . Then for any relatively compact, simply-connected polydomain $K \subset G$,

- (i) the sheaf $\mathcal{S}|_K$ is generated by finitely many global sections in $\mathcal{S}(K)$.
- (ii) $H^q(K, \mathcal{S}) = 0$ for all $q \geq 1$.

PROOF. By *Theorem 3.4.9*, \mathcal{S} admits a terminating chain of syzygies over an open neighbourhood of \overline{K} . Over K the syzygy takes the form

$$\cdots \longrightarrow (\mathcal{O}|_K)^{p_1} \xrightarrow{\mu_1} (\mathcal{O}|_K)^p \xrightarrow{\mu} \mathcal{S}|_K \longrightarrow 0. \quad (42)$$

Let $e_k \in \mathcal{O}^p(K)$ denote the canonical generators of $(\mathcal{O}|_K)^p$. By the exactness of (42), $\mathcal{S}|_K$ is generated by the global sections $\mu(e_k)$ and this proves (i). Moreover, since \mathcal{S} admits a terminating chain of syzygies in an open neighbourhood of \overline{K} , *Proposition 3.4.2* implies that $H^q(K, \mathcal{S}) = 0$ for all $q \geq 1$, and this proves (ii). \square

Cartan's Theorem B and its Consequences

In this chapter, we complete the proof of Cartan's Theorem B for Stein spaces. In §4.1 we construct a suitable Fréchet topology on the space of sections of a coherent analytic sheaf \mathcal{S} over an analytic block $S_\nu \subset X$, where X is a Stein space. This will allow us to approximate sections in $\mathcal{S}(X)$ by sections in $\mathcal{S}(S_\nu)$, c.f., *Remark 3.3.4*. In §4.2 the proof of Cartan's Theorem B is given, together with its converse. We conclude this chapter with a discussion of the Cousin problems in §4.3.

§4.1. CONSTRUCTING THE SEMINORMS AND THE FRÉCHET SPACE $\mathcal{S}(\mathring{A}(S))$

We remind the reader of the definition of a Fréchet space.

Definition 4.1.1. A complex topological vector space V with a Hausdorff topology is called a *Fréchet space* if its topology is determined by an at most countable family of seminorms $\rho_k : V \rightarrow \mathbb{R}$, and V is complete with respect to this topology.

Recall that a map $\rho : V \rightarrow \mathbb{R}$ is called a *seminorm* if

- (i) ρ is *non-negative*, i.e., $\rho(x) \geq 0$ for all $x \in V$.
- (ii) ρ is *homogeneous*, i.e., $\rho(\lambda x) = |\lambda| \rho(x)$ for all $\lambda \in \mathbb{C}$, $x \in V$.
- (iii) ρ satisfies the *triangle inequality*, i.e., $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in V$.

Remark 4.1.2. Fréchet spaces generalise the more familiar Banach spaces, where the norm is replaced with a countable family of seminorms. One may also readily show that a Fréchet space is *metrizable*, i.e., it is homeomorphic to a metric space. The distance function is defined by

$$d(x, y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\rho_k(x - y)}{1 + \rho_k(x - y)}.$$

The reader may wish to consult [37, p. 6] for further details.

Example 4.1.3. Given a domain $G \subseteq \mathbb{C}^n$, we will consider the set of analytic functions $f : G \rightarrow \mathbb{C}$ as a Fréchet space when equipped with the *topology of uniform convergence on compact subsets* (see, e.g., [32, p. 91–92] for details).

Notational Remark 4.1.4. We fix the following notation throughout this section: we let $X := (X, \mathcal{O}_X)$ denote a complex analytic space and \mathcal{S} a coherent analytic sheaf on X . We let $(S, \pi) := (S, \pi, Q, W)$ be an analytic block in X and, unless otherwise stated, U and V will denote respective open neighbourhoods of S and Q such that $\tau := \pi|_U : U \rightarrow V$ is a finite map.

The purpose of this section is to construct *good seminorms* $\|\cdot\|_Q$ on $\mathcal{S}(S)$ which will behave well under an exhaustion by analytic blocks. The existence (and uniqueness) of a Fréchet space structure on the space of sections of a coherent analytic sheaf is non-trivial. Unfortunately, in the interests of brevity, we cannot detail the rich theory behind this development. The interested reader may wish to consult [19, Chapter V.6] for details. In this section, we prove the following theorem:

Theorem 4.1.5. Let $(S_\nu, \pi_\nu)_{\nu \geq 1}$ be an exhaustion of X by analytic blocks. Then

- (i) every complex vector space $\mathcal{S}(S_\nu)$ may be endowed with seminorms $\|\cdot\|_\nu$ such that, with respect to this topology, the space of global sections $\mathcal{S}(X)$, when restricted to S_ν , is dense in $\mathcal{S}(S_\nu)$.
- (ii) the restriction maps $\text{res}_\nu : \mathcal{S}(S_{\nu+1}) \rightarrow \mathcal{S}(S_\nu)$ are bounded, i.e., for each $\nu \geq 1$, there exists a constant $M_\nu \in \mathbb{R}_{>0}$ such that $\|\text{res}_\nu(s)\|_\nu \leq M_\nu \|s\|_{\nu+1}$.
- (iii) if $(s'_j)_{j \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}(S_\nu)$, then for each $\nu \geq 2$, the restricted sequence $(\text{res}_{\nu-1}(s'_j))_{j \in \mathbb{N}}$ has a limit in $\mathcal{S}(S_{\nu-1})$.
- (iv) if $s \in \mathcal{S}(S_\nu)$ satisfies $\|s\|_\nu = 0$ then for each $\nu \geq 2$, $\text{res}_{\nu-1}(s) = 0$.

By *Theorem 3.1.32* the direct image $\mathcal{T} := \tau_*(\mathcal{S}|_U)$ is a coherent analytic sheaf on V . For some $\ell \geq 1$, part (i) of *Theorem 3.4.10* then informs us that there is a surjection of sheaves of analytic sheaves $\Phi_{Q,\ell} : \mathcal{O}^\ell|_Q \rightarrow \mathcal{T}|_Q$, and by part (ii) of *Theorem 3.4.10* this induces a surjection of $\mathcal{O}(Q)$ -modules, which we also write as $\Phi_{Q,\ell} : \mathcal{O}^\ell(Q) \rightarrow \mathcal{T}(Q)$. Observe that since $S = \tau^{-1}(Q)$, we have a canonical isomorphism of complex vector spaces $\varphi : \mathcal{S}(S) \rightarrow \mathcal{T}(Q)$. We therefore make the following definition:

Definition 4.1.6. Maintaining the above notation we define $\|\cdot\|_Q : \mathcal{S}(S) \rightarrow \mathbb{R}$ by

$$\|s\|_Q := \inf \left\{ \max_{z \in Q} |f(z)| : f \in \mathcal{O}^\ell(Q) \text{ and } \Phi_{Q,\ell}(f) = \varphi(s) \right\}. \quad (43)$$

A seminorm of this form is referred to as a *good seminorm*.

We will need the following result. The proof is omitted in the interests of brevity.

Lemma 4.1.7. ([19, p. 118]). If M is a complex manifold and \mathcal{J} is a coherent subsheaf of \mathcal{O}^ℓ on M , $1 \leq \ell < \infty$, then, with respect to the topology of uniform convergence on compact subsets, the module of sections $\mathcal{J}(M)$ is a closed vector subspace of $\mathcal{O}^\ell(X)$.

Proposition 4.1.8. The map $\|\cdot\|_Q : \mathcal{S}(S) \rightarrow \mathbb{R}$ defined in *Definition 4.1.6* is a seminorm on $\mathcal{S}(S)$. Moreover, if $\|s\|_Q = 0$ then the restriction of s to the analytic interior $\mathring{A}(S)$ vanishes.

PROOF. The fact that (43) defines a seminorm is clear, and is essentially deduced from the fact that $f \mapsto \max_{z \in Q} |f(z)|$ satisfies the requirements of a seminorm¹. The interesting part of the proposition is therefore the second statement. Suppose that $\|s\|_Q = 0$. Then there exists a sequence $(f_j)_{j \in \mathbb{N}}$ such that $\Phi_{Q,\ell}(f_j) = \varphi(s)$ and $\lim_{j \rightarrow \infty} \max_{z \in Q} |f_j(z)| = 0$. Set $h := f_1 \in \mathcal{O}^\ell$, and $h_j := h - f_j \in \ker(\Phi_{Q,\ell})$ for each $j \in \mathbb{N}$. Then $\lim_{j \rightarrow \infty} \max_{z \in Q} |h - h_j| = 0$, and in particular, $\lim_{j \rightarrow \infty} (h_j|_Q) = h|_Q$ with respect to the

¹One may also see this from the fact that (43) is simply the quotient seminorm on $\mathcal{T}(Q) = \mathcal{O}^\ell / \ker \Phi_{Q,\ell}$. This observation provides a clear insight as to why (43) does not define a norm. Indeed, if (43) is a norm then $\ker \Phi_{Q,\ell}$ is a closed subspace of \mathcal{O}^ℓ , but this is of course not true in general.

topology of uniform convergence on compact sets. Since $\mathcal{K}er(\Phi_{Q,\ell})$ is a coherent subsheaf of $\mathcal{O}^\ell|_Q$, *Lemma 4.1.7* informs us that $\Phi_{Q,\ell}(h|_{\mathring{Q}}) = 0$. Therefore, $\varphi(s)|_{\mathring{Q}} = \Phi_{Q,\ell}(h|_Q) = 0$, and since $\mathring{A}(S) = \tau^{-1}(\mathring{Q})$, it follows that the restriction of s to $\mathring{A}(S)$ is zero, as required. \square

We will use the seminorm $\|\cdot\|_Q$ to define a distance on $\mathcal{S}(\mathring{A}(S))$. The above notation will be maintained.

Construction 4.1.9. Let $(Q_\nu)_{\nu \geq 1}$ be an exhaustion of $\mathring{Q} \subset \mathbb{C}^m$ by compact (Euclidean) blocks, with each block having the same centre. That is, for each $\nu \geq 1$, $Q_\nu \subset \mathring{Q}_{\nu+1}$, and $\mathring{Q} = \bigcup_{\nu \geq 1} Q_\nu$. Then for each $\nu \geq 1$, the restriction of $\Phi_{Q,\ell}$ to Q_ν induces a surjection of $\mathcal{O}(Q_\nu)$ -modules $\Phi_{Q_\nu,\ell} : \mathcal{O}^\ell(Q_\nu) \rightarrow \pi_*(\mathcal{S}|_S)(Q_\nu)$, and we obtain a seminorm $\|\cdot\|_{Q_\nu} : \mathcal{S}(\tau^{-1}(Q_\nu)) \rightarrow \mathbb{R}$ specified by the appropriate change of the formula (43).

Notational Remark 4.1.10. To avoid cumbersome notation we will write $\|\cdot\|_\nu$ in place of $\|\cdot\|_{Q_\nu}$.

Definition 4.1.11. We define a function $d : \mathcal{S}(\mathring{A}(S)) \times \mathcal{S}(\mathring{A}(S)) \rightarrow \mathbb{R}$ by

$$d(s_1, s_2) := \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \cdot \frac{\|s_1 - s_2\|_\nu}{1 + \|s_1 - s_2\|_\nu}.$$

Observation 4.1.12. Let us make the observation that if $K \subset X$ is a compact set, then $\mathcal{O}^\ell(K)$ is not a good space to work with. Indeed, for K compact, $\mathcal{O}^\ell(K)$ is normable, i.e., it is equipped with a norm, but it is not complete with respect to this norm. If $K \subset X$ is a domain, however, then $\mathcal{O}^\ell(K)$ is not normable, i.e., it cannot be equipped with a norm, but it is complete.

Lemma 4.1.13. The function d defined in *Definition 4.1.6* is a distance. Endowing $\mathcal{S}(\mathring{A}(S))$ with this distance d , we obtain a Fréchet space $(\mathcal{S}(\mathring{A}(S)), d)$.

PROOF. The fact that d defines a distance follows from *Proposition 4.1.8* and *Remark 4.1.2*. We need to show d is a complete distance on $\mathcal{S}(\mathring{A}(S))$. To this end, let $(s_\mu)_{\mu \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{S}(\mathring{A}(S))$, and let $Q \subset \mathbb{C}^m$ be the compact (Euclidean) block associated to S . Exhaust \mathring{Q} by compact (Euclidean) blocks $(Q_\nu)_{\nu \in \mathbb{N}}$, each with the same centre. For each ν , we may choose a bounded sequence $(f_\nu^j)_{j \in \mathbb{N}} \in \mathcal{O}^\ell(Q_\nu)$, such that $\Phi_{Q_\nu,\ell}(f_\nu^j) = s_j|_{\pi^{-1}(Q_\nu) \cap S}$. By Montel's theorem, there is a subsequence (also denoted by) $(f_\nu^j)_{j \in \mathbb{N}}$, which converges uniformly on $Q_{\nu-1}$ to some $f_\nu \in \mathcal{O}^\ell(Q_{\nu-1})$. Since $\Phi_{Q_{\nu-1},\ell}(f_{\nu+1})|_{\mathring{Q}_{\nu-1}} = \Phi_{Q_{\nu-1},\ell}(f_\nu)|_{\mathring{Q}_{\nu-1}}$, we may glue these f_ν together to obtain a global section $f \in \mathcal{O}^\ell(Q)$ such that $\Phi_{Q,\ell}(f)|_{Q_\nu} = \Phi_{Q_\nu,\ell}(f_\nu) = s_j|_{\pi^{-1}(Q_\nu) \cap S}$. By then setting $s := \Phi_{Q,\ell}(f)$, we obtain the desired limit, and so $\mathcal{S}(\mathring{A}(S))$ is complete, as required. \square

Note that the distance d constructed from the good seminorms $\|\cdot\|_\nu$ required two choices: we needed to choose an exhaustion $(Q_\nu)_{\nu \geq 1}$ of \mathring{Q} and also choose a surjection $\Phi_{Q,\ell}$. The reader will be reassured to know that this topology is independent of both of these choices. The following lemma informs us that this topology is not dependent on the choice of exhaustion, surjection, and is also not changed if (S, π) is replaced with (S, π') , where $\pi' := (\pi, \varphi)$, for some holomorphic map $\varphi : X \rightarrow \mathbb{C}^n$. We first set up some notation:

Let (S, π) be an analytic block in X and let $\varphi : X \rightarrow \mathbb{C}^n$ be a holomorphic map such that $\varphi(S)$ is contained in some compact (Euclidean) block $Q^* \subset \mathbb{C}^n$. Set $\pi' := (\pi, \varphi) : X \rightarrow \mathbb{C}^m \times \mathbb{C}^n$ and $\mathbb{C}'^m := \mathbb{C}^m \times \mathbb{C}^n$, and observe that $S = \pi'^{-1}(Q \times Q^*) \cap U$. Now choose open neighbourhoods $U' \subset X$ and $V' \subset \mathbb{C}'^m$ of S and $Q \times Q^*$

respectively, such that $'\pi|_U : 'U \rightarrow 'V$ is finite. Fix an surjection of sheaves $'\Phi_{Q,\ell} : \mathcal{O}'^\ell|_{Q \times Q^*} \rightarrow '\pi_*(\mathcal{S}|_S)$ and exhaust $Q \times Q^*$ by the open sets $(\mathring{Q}_\nu \times \mathring{Q}_\nu^*)_{\nu \geq 1}$. We denote by $'d$ be the associated distance on $\mathcal{S}(\mathring{A}(S))$.

Proposition 4.1.14. The topologies on $\mathcal{S}(\mathring{A}(S))$ induced by the distances d and $'d$ coincide.

PROOF. Let e_1, \dots, e_ℓ be the canonical generators of $\mathcal{O}^\ell(Q)$. For each $1 \leq k \leq \ell$, set $'e_k := '\Phi_{Q,\ell}^{-1} \circ \Phi_{Q,\ell}(e_k) \in \mathcal{O}'^\ell(Q \times Q^*)$ to be the preimage of $\Phi_{Q,\ell}(e_k) \in \pi_*(\mathcal{S}|_S)(Q) = '\pi_*(\mathcal{S}|_S)(Q \times Q^*)$ in $\mathcal{O}'^\ell(Q \times Q^*)$ under $'\Phi_{Q,\ell}$. For $f \in \mathcal{S}(Q_\nu)$, let $'f \in \mathcal{O}(Q_\nu \times Q_\nu^*)$ denote the holomorphic extension of f to $Q_\nu \times Q_\nu^*$, constant along each of the fibres $\{q\} \times Q_\nu^*$, $q \in Q_\nu$. For each $\nu \geq 1$, we define \mathbb{C} -linear operators $\mathcal{O}^\ell(Q_\nu) \rightarrow \mathcal{O}'^\ell(Q_\nu \times Q_\nu^*)$,

$$\sum_{k=1}^{\ell} f_k \cdot e_k \mapsto \sum_{k=1}^{\ell} 'f_k \cdot 'e_k.$$

The norms of these operators are bounded by a constant which does not depend on ν . In particular, the identity map $\text{Id} : (\mathcal{S}(\mathring{A}(S)), d) \rightarrow (\mathcal{S}(\mathring{A}(S)), 'd)$ is continuous, and the Fréchet open mapping theorem² informs us that this map is a homeomorphism. In other words, the topologies induced from the distances d and $'d$ coincide, as required. \square

Lemma 4.1.15. Let $(S_1, \pi_1) \preceq (S_2, \pi_2)$ be an inclusion of analytic blocks in X . There exists an analytic block (S, π_1) such that $(S_1, \pi_1) \preceq (S, \pi_1) \preceq (S_2, \pi_2)$.

PROOF. Recall that by $(S_1, \pi_1) \preceq (S_2, \pi_2)$ it is understood that $\mathbb{C}^{m_2} := \mathbb{C}^{m_1} \times \mathbb{C}^n$ for some $n \in \mathbb{N}$, $\pi_2 = (\pi_1, \varphi)$ for some holomorphic map $\varphi : X \rightarrow \mathbb{C}^n$, and there exists a point $q \in \mathbb{C}^n$ such that $Q_1 \times \{q\} \subset Q_2$. Set $Q' := Q_2 \cap (\mathbb{C}^{m_1} \times \{q\}) \subset \mathbb{C}^{m_1}$ and denote by Q^* the image of Q_2 under the projection $\mathbb{C}^{m_2} \rightarrow \mathbb{C}^n$. We choose open neighbourhoods $U \subset \mathring{A}(S_2)$ of S_1 and $V \subset Q'$ of Q_1 such that $\pi_1|_U : U \rightarrow V$ is finite, and $\pi_1^{-1}(Q_1) \cap U = S_1$. Choose a compact (Euclidean) block $Q \subset \mathbb{C}^{m_1}$ such that $Q_1 \subset \mathring{Q} \subset Q \subset \mathring{Q}'$. We then obtain the desired analytic block (S, π_1) by setting $S := \pi_1^{-1}(Q) \cap U$. \square

Proposition 4.1.16. Let $(S_1, \pi_1) \preceq (S_2, \pi_2)$ be an inclusion of analytic blocks in X . Then the restriction map $\rho : \mathcal{S}(S_2) \rightarrow \mathcal{S}(S_1)$ is bounded. In other words, we may find a constant $M \in \mathbb{R}$ such that $\|\rho(s)\|_1 \leq M\|s\|_2$.

PROOF. The decomposition in *Lemma 4.1.15* allows us to split the map $\rho : \mathcal{S}(S_2) \rightarrow \mathcal{S}(S_1)$ as the composition of $\rho_1 : \mathcal{S}(S_2) \rightarrow \mathcal{S}(\mathring{A}(S))$ and $\rho_2 : \mathcal{S}(\mathring{A}(S)) \rightarrow \mathcal{S}(S_1)$, where the maps are the obvious restrictions. If $\mathring{A}(S)$ is equipped with the topology defined by π_2 , the map $\mathcal{S}(S_2) \rightarrow \mathcal{S}(\mathring{A}(S))$ is continuous. Similarly, if $\mathring{A}(S)$ is equipped with the topology defined by π_1 , the map $\mathcal{S}(\mathring{A}(S)) \rightarrow \mathcal{S}(S_1)$ is continuous. By *Proposition 4.1.14* these topologies coincide, and therefore the composition $\rho = \rho_2 \circ \rho_1$ is continuous and in particular, ρ is bounded. \square

²The Fréchet open mapping theorem states that if X and Y are Fréchet spaces and $T : X \rightarrow Y$ is a continuous surjective linear operator, T is an open map (see [63, Theorem 1.6]).

Proposition 4.1.17. Let $(S_1, \pi_1) \preceq (S_2, \pi_2)$ be an inclusion of analytic blocks in X . For every Cauchy sequence $(s_j)_{j \in \mathbb{N}}$ in $\mathcal{S}(S_2)$, the restriction of s_j to $\mathcal{S}(S_1)$ converges to a unique limit $s \in \mathcal{S}(S_1)$.

PROOF. Let (S, π_1) be the intermediate analytic block from *Lemma 4.1.15* and decompose $\rho : \mathcal{S}(S_2) \rightarrow \mathcal{S}(S_1)$ as in the proof of *Proposition 4.1.16*. Let $(s_j)_{j \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{S}(S_2)$. The map $\rho_1 : \mathcal{S}(S_2) \rightarrow \mathcal{S}(\mathring{A}(S))$ is uniformly continuous, so $(\rho_1(s_j))_{j \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}(\mathring{A}(S))$. Since $\mathcal{S}(\mathring{A}(S))$ is complete, $(\rho_1(s_j))_{j \in \mathbb{N}}$ converges to a unique limit $t \in \mathcal{S}(\mathring{A}(S))$, and by the continuity of $\rho_2 : \mathcal{S}(\mathring{A}(S)) \rightarrow \mathcal{S}(S_1)$, the image of $(\rho_1(s_j))_{j \in \mathbb{N}}$ converges to the unique limit $\rho_2(t) \in \mathcal{S}(S_1)$; setting $s = \rho_2(t)$ completes the proof. \square

All that remains is to prove statement (i) of *Theorem 4.1.5*. The proof of statement (i) requires a theorem of Runge type for compact (Euclidean) blocks.

Notational Remark 4.1.18. For $z = x + iy$, we set

$$R := \{z \in \mathbb{C} : a \leq x \leq b, c \leq y \leq d\}.$$

Let K' be an arbitrary non-empty compact set in \mathbb{C}^{n-1} . The set $K := R \times K'$ is a compact set in $\mathbb{C}^m = \mathbb{C} \times \mathbb{C}^{m-1}$. Let z denote the coordinate of \mathbb{C} and $z' = (z_2, \dots, z_m)$ denote the coordinates for \mathbb{C}^{m-1} in the decomposition $\mathbb{C}^m = \mathbb{C} \times \mathbb{C}^{m-1}$, we also set $w = (z, z') \in \mathbb{C}^m$.

Proposition 4.1.19. (*Runge*). For any $\varepsilon > 0$ and $f \in \mathcal{O}(K)$ there exists a polynomial $p(w)$, with coefficients holomorphic on K' , such that $\sup_{w \in K} |f(w) - p(w)| < \varepsilon$.

PROOF. Assume that $R \neq \emptyset$. Choose an open rectangle E containing R and a function $\tilde{f} \in \mathcal{O}(\bar{E} \times K')$ such that $\tilde{f}|_K = f$. Denote by ∂E the oriented boundary of E . By the Cauchy integral formula of one complex variable, for all $w = (z, z') \in K$,

$$f(w) = \frac{1}{2\pi i} \int_{\partial E} \frac{f(\zeta, z')}{\zeta - z} d\zeta. \quad (44)$$

The function $\lambda(\zeta, w) = (2\pi i(\zeta - z))^{-1} f(\zeta, z')$ in the integrand of (44) is uniformly continuous on the compact set $\partial E \times K$. In particular, for any $\varepsilon > 0$, we may choose $\delta > 0$ such that $|\zeta - \zeta'| < \delta$ implies $|\lambda(\zeta', w) - \lambda(\zeta, w)| \leq \frac{1}{2L} \varepsilon$, where L denotes the circumference of E .

Now partition ∂E into n intervals I_ν , $1 \leq \nu \leq n$, each of which has length $\delta_\nu < \delta$, and for each interval I_ν , choose a point $\zeta_\nu \in I_\nu$. Then $\lambda(\zeta_\nu, w)$ is holomorphic on $(\mathbb{C} \setminus \{\zeta_\nu\}) \times K$, and we can approximate (44) by the

Riemann sum $g(w) := \sum_{\nu=1}^n \lambda(\zeta_\nu, w) \delta_\nu$. Observe that

$$\begin{aligned}
\sup_{w \in K} |f(w) - g(w)| &= \sup_{w \in K} \left| \frac{1}{2\pi i} \int_{\partial E} \frac{f(\zeta, z')}{\zeta - z} d\zeta - \sum_{\nu=1}^n \lambda(\zeta_\nu, w) \delta_\nu \right| \\
&= \sup_{w \in K} \left| \sum_{\nu=1}^n \int_{I_\nu} [\lambda(\zeta, w) - \lambda(\zeta_\nu, w)] d\zeta \right| \\
&\leq \sum_{\nu=1}^n \int_{I_\nu} \sup_{w \in K} |\lambda(\zeta, w) - \lambda(\zeta_\nu, w)| d\zeta \\
&\leq \sum_{\nu=1}^n \frac{\varepsilon}{2L} \delta_\nu = \frac{\varepsilon}{2}.
\end{aligned} \tag{45}$$

For each point $\zeta_\nu \in I_\nu$, choose a disk $\Delta_\nu \subset \mathbb{C}$ such that $R \subset \Delta_\nu$ and $\zeta_\nu \notin \overline{\Delta_\nu}$. Let $T_\nu \in \mathbb{C}[z]$ denote the Taylor polynomial obtained from the Taylor expansion of $(2\pi i(\zeta_\nu - z))^{-1} \delta_\nu$ expanded about the centre of Δ_ν such that

$$\sup_{z \in R} |(2\pi i(\zeta_\nu - z))^{-1} \delta_\nu - T_\nu(z)| \leq \frac{1}{2nLM_\nu} \varepsilon, \tag{46}$$

where $M_\nu := \max_{z' \in K'} |f(\zeta_\nu, z')|$. The function $p(w) := \sum_{\nu=1}^n \tilde{f}(\zeta_\nu, z') T_\nu(z)$ is then a polynomial in z with coefficients holomorphic on K' . Then from (45) and (46) we have the estimate

$$\begin{aligned}
\sup_{w \in K} |f(w) - p(w)| &\leq \sup_{w \in K} |f(w) - g(w)| + \sup_{w \in K} |g(w) - p(w)| \\
&\stackrel{(45)}{\leq} \frac{\varepsilon}{2} + \sup_{w \in K} |g(w) - p(w)| \\
&= \frac{\varepsilon}{2} + \sup_{w \in K} \left| \sum_{\nu=1}^n \lambda(\zeta_\nu, z') \delta_\nu - \sum_{\nu=1}^n f(\zeta_\nu, z') T_\nu(z) \right| \\
&\leq \frac{\varepsilon}{2} + \sum_{\nu=1}^n \sup_{z' \in K'} |f(\zeta_\nu, z')| \cdot \sup_{z \in R} |(2\pi i(\zeta_\nu - z))^{-1} - T_\nu(z)| \\
&\stackrel{(46)}{\leq} \frac{\varepsilon}{2} + \sum_{\nu=1}^n M_\nu \cdot \frac{1}{2nLM_\nu} \varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

as required. \square

Corollary 4.1.20. Given $\varepsilon > 0$ and any $f \in \mathcal{O}(Q)$, where $Q \subset \mathbb{C}^m$ is a compact block, there exists a polynomial $p(z)$ such that $\sup_{z \in Q} |f(z) - p(z)| < \varepsilon$.

PROOF. We proceed by induction on m . For $m = 1$, the compact block $Q \subset \mathbb{C}$ is the rectangle R of *Proposition 4.1.19*. In this case, the compact set K' is a point, and the polynomial with coefficients holomorphic on K' is just a constant coefficient polynomial. Now suppose that $n > 1$. For $Q \subset \mathbb{C}^m$, we write $Q = R \times Q'$, where $R \subset \mathbb{C}$ is a compact rectangle, and $Q' \subset \mathbb{C}^{m-1}$ is a compact block. Applying *Proposition 4.1.19*, we find a polynomial $\tilde{p}(w) = \tilde{p}(z, z') := \sum_{\nu=0}^n f_\nu(z') z^\nu$, where $f_\nu \in \mathcal{O}(Q')$, such that $\sup_{w \in Q} |f(w) - \tilde{p}(w)| < \frac{1}{2} \varepsilon$. The induction hypothesis informs us of the existence of polynomials $g_\nu \in \mathbb{C}[z']$ such that $\sup_{z' \in Q'} |f_\nu(z') - g_\nu(z')| \leq \frac{\varepsilon}{2(n+1)N}$, where $N := \max_{0 \leq \nu \leq n} \sup_{z \in R} |z^\nu|$. Setting

$p(w) = \sum_{\nu=0}^n g_\nu(z')z^\nu$, we see that

$$\begin{aligned}
\sup_{w \in Q} |f(w) - p(w)| &\leq \sup_{w \in Q} |f(w) - \tilde{p}(w)| + \sup_{w \in Q} |\tilde{p}(w) - p(w)| \\
&\leq \frac{\varepsilon}{2} + \sup_{w \in Q} \left| \sum_{\nu=0}^n f_\nu(z')z^\nu - \sum_{\nu=0}^n g_\nu(z')z^\nu \right| \\
&\leq \frac{\varepsilon}{2} + \sum_{\nu=0}^n \sup_{w \in Q} |f_\nu(z') - g_\nu(z')| \cdot \max_{0 \leq \nu \leq n} \left(\sup_{z \in R} |z^\nu| \right) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2(n+1)N} \cdot N \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

as required. \square

Now for an inclusion of analytic blocks $(S_1, \pi_1) \preceq (S_2, \pi_2)$, *Lemma 4.1.15* provides us an intermediate analytic block (S, π_1) . Let U_2 and V_2 be the open neighbourhoods of S_2 and Q_2 respectively, such that $\pi_2|_{U_2} : U_2 \rightarrow V_2$ is finite. The surjective morphism of sheaves $\Phi_{Q_2, \ell_2} : \mathcal{O}^{\ell_2}|_{Q_2} \rightarrow (\pi_2)_*(\mathcal{S}|_{U_2})|_{Q_2}$ induces a map $\alpha_2 : \mathcal{O}^{\ell_2}(Q_2) \rightarrow \mathcal{S}(S_2)$. The restriction of Φ_{Q_2, ℓ_2} to the intermediate block S yields a map $\Phi_{Q, \ell_2} : \mathcal{O}^{\ell_2}|_Q \rightarrow (\pi_2)_*(\mathcal{S}|_{U_2})|_Q$ and since $(\pi_2)_*(\mathcal{S}|_{U_2})|_Q \cong \mathcal{S}(S_1)$, the map Φ_{Q, ℓ_2} determines a \mathbb{C} -linear surjection $\alpha : \mathcal{O}^{\ell_2}(Q) \rightarrow \mathcal{S}(S)$ together with an associated seminorm $\|\cdot\|_S$ on $\mathcal{S}(S)$.

Lemma 4.1.21. Maintain the notation in the above paragraph. The restriction of $\mathcal{S}(S_2)$ to S is dense in $\mathcal{S}(S)$.

PROOF. By construction, we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{O}^{\ell_2}(Q_2) & \xrightarrow{\alpha_2} & \mathcal{S}(S_2) \\
\text{res} \downarrow & & \downarrow \rho_1 \\
\mathcal{O}^{\ell_2}(Q) & \xrightarrow{\alpha} & \mathcal{S}(S)
\end{array}$$

where the vertical maps are given by restriction. By *Corollary 4.1.20*, $\mathcal{O}^{\ell_2}(Q_2)$ is dense in $\mathcal{O}^{\ell_2}(Q)$. Therefore, since all maps in the above diagram are continuous, $\mathcal{S}(S_2)|_S$ is dense in $\mathcal{S}(S)$ as claimed. \square

We now show that, for an inclusion of analytic blocks $(S_1, \pi_1) \preceq (S_2, \pi_2)$, we may choose the intermediate analytic block (S, π_1) such that the restriction map $\rho_2 : \mathcal{S}(S) \rightarrow \mathcal{S}(S_1)$ is a surjection.

Lemma 4.1.22. Let $(S_1, \pi_1) \preceq (S_2, \pi_2)$ in X , and let S be the intermediate analytic block from *Lemma 4.1.15*. There exists a compact set $\tilde{S} \subset X$ such that $\tilde{S} \cap S_1 = \emptyset$ and $S = S_1 \cup \tilde{S}$.

PROOF. Let $\varphi : X \rightarrow \mathbb{C}^n$ be the holomorphic map in the definition of an inclusion of analytic blocks, and let $Q := (Q_1 \times \mathbb{C}^n) \cap Q_2$. Then from the proof of *Lemma 4.1.15*, we see that the intermediate block is given by $S = \pi_1^{-1}(Q_1) \cap S_2$. Since $S_1 \subset S_2$ and $S_1 \subset \pi_1^{-1}(Q_1)$, we see that $S_1 \subset S$. Moreover, we may choose an open neighbourhood U of S_1 such that $S_1 = \pi_1^{-1}(Q_1) \cap U$ and in particular the set $\tilde{S} := S \setminus S_1$ is compact. \square

Let us note that the space of sections of a coherent sheaf on a compact set has no good topology. For example, the space of holomorphic functions on the closed disk is a normed space, but is not complete. The Fréchet topology considered here is defined for domains, and we must therefore extend the compact (Euclidean) blocks Q_1 , Q , and Q_2 , to domains \widehat{Q}_1 , \widehat{Q} , and \widehat{Q}_2 . Similarly, S_1 , S , and S_2 are extended to \widehat{S}_1 , \widehat{S} , and \widehat{S}_2 . This extension is carried out in such a way that $\widetilde{S} \cap \widehat{S}_1 = \emptyset$. Moreover, by *Lemma 4.1.13*, $\mathcal{S}(\mathring{A}(\widehat{S}))$ and $\mathcal{S}(\mathring{A}(\widehat{S}_1))$ carry Fréchet topologies such that the restriction map $\mathcal{S}(\mathring{A}(\widehat{S})) \rightarrow \mathcal{S}(\mathring{A}(\widehat{S}_1))$ is a surjective continuous map (see also [19, p. 164]).

Lemma 4.1.23. Let $(S_1, \pi_1) \preceq (S_2, \pi_2)$ be an inclusion of analytic blocks in X . Then $\mathcal{S}(S_2)|_{S_1}$ is dense in $\mathcal{S}(S_1)$.

PROOF. Fix a section $s \in \mathcal{S}(S_1)$. The map $\rho_2 : \mathcal{S}(S) \rightarrow \mathcal{S}(S_1)$ is surjective, so there exists a section $s_1 \in \mathcal{S}(S)$ such that $\rho_2(s_1) = s_1|_{S_1} = s$. Extending the blocks in the manner described in the preceding paragraph, we can extend s_1 to a section $\widehat{s}_1 \in \mathcal{S}(\widehat{S})$. Now there exists a sequence $(s_j)_{j \in \mathbb{N}} \subset \mathcal{S}(\widehat{S}_2)$ such that $s_j|_{\widehat{S}} \rightarrow \widehat{s}_1$ in $\mathcal{S}(\widehat{S})$. The restriction map $\mathcal{S}(\mathring{A}(\widehat{S})) \rightarrow \mathcal{S}(\mathring{A}(\widehat{S}_1))$ is continuous in the Fréchet topology, so $s_j|_{\mathring{A}(\widehat{S}_1)} \rightarrow \widehat{s}_1|_{\mathring{A}(\widehat{S}_1)}$ in $\mathcal{S}(\mathring{A}(\widehat{S}_1))$. From the definition of a good seminorm however, we see that $s_j|_{S_1} \rightarrow \widehat{s}_1|_{S_1} = s_1|_{S_1} = s$, as required. \square

Proposition 4.1.24. Let $(S_\nu, \pi_\nu)_{\nu \geq 1}$ be an exhaustion of X by analytic blocks. For each $\nu \geq 1$, the restriction of $\mathcal{S}(X)$ to S_ν is dense in $\mathcal{S}(S_\nu)$.

PROOF. It suffices by *Lemma 4.1.23* to show that $\mathcal{S}(X)|_{S_1}$ is dense in $\mathcal{S}(S_1)$. Let $s \in \mathcal{S}(S_1)$, $\delta \in \mathbb{R}_{>0}$, and choose a sequence $(\delta_\nu)_{\nu \geq 1} \in \mathbb{R}_{>0}$ such that $\sum_{\nu \geq 1} \delta_\nu < \delta$. By *Lemma 4.1.23* we may construct a sequence $s_\nu \in \mathcal{S}(S_\nu)$ such that $s_1 = s$, and for each $\nu \geq 1$, $\|\text{res}_\nu(s_{\nu+1}) - s_\nu\|_\nu < \delta_\nu$. The sequence $(\text{res}_{\nu+1}(s_\mu))_{\mu > \nu}$ is then a Cauchy sequence in $\mathcal{S}(S_{\nu+1})$, and by *Proposition 4.1.14* this sequence has a unique limit $t_\nu \in \mathcal{S}(S_\nu)$. By *Proposition 4.1.16* each restriction map is bounded, and $\text{res}_\nu(t_{\nu+1})$ is also a limit of the sequence $\text{res}_\nu(s_\mu)$. By *Proposition 4.1.8* these limits t_ν and $t_{\nu+1}$ are equal on $\mathring{A}(S_\nu)$. Since X is exhausted by the $\mathring{A}(S_\nu)$, we may glue together each t_ν to obtain a global section $t \in \mathcal{S}(X)$ such that $\text{res}_\nu(t) = t_\nu$ for each $\nu \geq 1$. We now simply consider the estimate

$$\begin{aligned} \|\text{res}_1(t) - s\|_1 &= \left\| t_1 - \text{res}_1(s_\mu) + \sum_{\nu=1}^{\mu-1} (\text{res}_1(s_{\nu+1}) - \text{res}_1(s_\nu)) \right\|_1 \\ &\leq \|t_1 - \text{res}_1(s_\mu)\|_1 + \sum_{\nu=1}^{\mu-1} \|\text{res}_1(s_{\nu+1}) - \text{res}_1(s_\nu)\|_1 \\ &< \|t_1 - \text{res}_1(s_\mu)\|_1 + \sum_{\mu=1}^{\nu-1} \delta_\mu \rightarrow \delta, \end{aligned}$$

as $\mu \rightarrow \infty$, and this completes the proof. \square

§4.2. CARTAN'S THEOREM B FOR STEIN SPACES

In this section, we complete the proof of Cartan's theorem B for Stein spaces. We remind the reader of the statement.

Theorem 4.2.1. Let X be a Stein space, and \mathcal{S} a coherent analytic sheaf on X . Then for each $q \geq 1$, $H^q(X, \mathcal{S}) = 0$.

PROOF. By *Proposition 3.3.16* we may exhaust X by analytic blocks $(S_\nu, \pi_\nu)_{\nu \geq 1}$. In the same manner as the proof of *Proposition 3.3.3*, we have the commutative diagram

$$\begin{array}{cccccccccccc}
 0 & \longrightarrow & \mathcal{S}(X) & \xrightarrow{\iota} & \mathcal{S}^0(X) & \xrightarrow{d_0} & \mathcal{S}^1(X) & \xrightarrow{d_1} & \cdots & \longrightarrow & \mathcal{S}^{q-1}(X) & \xrightarrow{d_{q-1}} & \mathcal{S}^q(X) & \longrightarrow & \cdots \\
 & & \downarrow \text{res}_\nu & & \downarrow \text{res}_\nu & & \downarrow \text{res}_\nu & & \downarrow \text{res}_\nu & & \downarrow \text{res}_\nu & & \downarrow \text{res}_\nu & & \downarrow \text{res}_\nu \\
 0 & \longrightarrow & \mathcal{S}(S_\nu) & \xrightarrow{\iota} & \mathcal{S}^0(S_\nu) & \xrightarrow{d_0^\nu} & \mathcal{S}^1(S_\nu) & \xrightarrow{d_1^\nu} & \cdots & \longrightarrow & \mathcal{S}^{q-1}(S_\nu) & \xrightarrow{d_{q-1}^\nu} & \mathcal{S}^q(S_\nu) & \longrightarrow & \cdots
 \end{array}$$

Via the inclusion map ι , we identify $\mathcal{S}(X)$ with a subset of $\mathcal{S}^0(X)$. By *Proposition 3.3.3*, $H^q(X, \mathcal{S}) = 0$ for all $q \geq 2$, so we need only show that $H^1(X, \mathcal{S}) = 0$. That is, we need to show that for any $\alpha \in \ker d_0$, we may find some $\beta \in \mathcal{S}^0(X)$ such that $d_0(\beta) = \alpha$. We construct two sequences $\beta_\nu \in \mathcal{S}^0(S_\nu)$, $\delta_\nu \in \mathcal{S}(S_\nu) = \ker(d_0^\nu)$ such that

- (i) $d_0^\nu(\beta_\nu) = \text{res}_\nu(\alpha)$,
- (ii) $\text{res}_{\nu-1}(\beta_{\nu+1} + \delta_{\nu+1}) = \text{res}_{\nu-1}(\beta_\nu + \delta_\nu)$.

From the sequence $(\beta_{\nu+1} + \delta_{\nu+1})_{\nu \geq 1}$ we construct a global section $\beta \in \mathcal{S}^0(X)$ such that $\text{res}_\nu(\beta) = \text{res}_\nu(\beta_{\nu+1} + \delta_{\nu+1})$ and for each $\nu \geq 1$, $\text{res}_\nu(d_0(\beta)) = d_0^\nu(\text{res}_\nu(\beta_{\nu+1})) + d_0^\nu(\delta_{\nu+1}) = \text{res}_\nu(\alpha)$, i.e., $d_0(\beta) = \alpha$.

Let us proceed by induction to construct these sequences. Let $\beta'_\nu \in \mathcal{S}(S_\nu)$ denote the sequence such that $d_0^\nu(\beta'_\nu) = \text{res}_\nu(\alpha)$, given to us by *Proposition 3.3.14*. Set $\beta_1 = \beta'_1$, and assume that $\beta_1, \dots, \beta_\nu$ have been chosen to satisfy (i). Let $\gamma'_\nu := \text{res}_\nu(\beta'_{\nu+1}) - \beta_\nu$. We observe that

$$\begin{aligned}
 d_0^\nu(\gamma'_\nu) &= d_0^\nu(\text{res}_\nu(\beta'_{\nu+1}) - \beta_\nu) \\
 &= \text{res}_\nu(d_0^{\nu+1}(\beta'_{\nu+1})) - d_0^\nu(\beta_\nu) \\
 &= \text{res}_\nu(\alpha) - \text{res}_\nu(\alpha) = 0,
 \end{aligned} \tag{47}$$

where (47) follows from the commutativity of the above diagram, and so $\gamma'_\nu \in \ker d_0^\nu = \mathcal{S}(S_\nu)$. By *Proposition 4.1.24*, for each $\nu \geq 1$, we can choose a global section $\gamma_\nu \in \mathcal{S}(X)$ such that

$$\|\gamma'_\nu - \text{res}_\nu(\gamma_\nu)\|_\nu \leq 2^{-\nu}. \tag{48}$$

Set $\beta_{\nu+1} := \beta'_{\nu+1} - \text{res}_{\nu+1}(\gamma_\nu) \in \mathcal{S}^0(S_{\nu+1})$. Then

$$\begin{aligned}
 d_0^{\nu+1}(\beta_{\nu+1}) &= d_0^{\nu+1}(\beta'_{\nu+1}) - d_0^{\nu+1}(\text{res}_{\nu+1}(\gamma_\nu)) \\
 &= \text{res}_{\nu+1}(\alpha) - \text{res}_{\nu+1}(d_0(\gamma_\nu))
 \end{aligned} \tag{49}$$

$$= \text{res}_{\nu+1}(\alpha), \tag{50}$$

where (49) follows from the commutativity of the above diagram, and (50) follows from the fact that $\mathcal{S}(X) = \ker(d_0)$. To construct the sequence $\delta_\nu \in \mathcal{S}(S_{\nu-1})$, we define, for each $\nu \geq 1$, sequences $(s_j^\nu)_{j \in \mathbb{N}} \subset \mathcal{S}(S_{\nu-1})$,

$$s_j^\nu := \text{res}_\nu(\beta_{\nu+j}) - \beta_\nu.$$

We claim that these sequences are Cauchy in the topology induced from the seminorms $\|\cdot\|_\nu$. Assuming this claim for the moment, observe that by *Proposition 4.1.17* each restricted sequence $\text{res}_{\nu-1}(s'_j) \in \mathcal{S}(S_{\nu-1})$ has a limit $\delta_\nu \in \mathcal{S}(S_{\nu-1})$. By *Proposition 4.1.16* the restriction maps are bounded, and therefore $\text{res}_{\nu-1}(s'_j)^{\nu+1}$ converges to $\text{res}_{\nu-1}(\delta_{\nu+1})$ as $j \rightarrow \infty$. We observe that $s'_j - \text{res}_\nu(s'_{j-1})^{\nu+1} = \text{res}(\beta_{\nu+1}) - \beta_\nu$, and since $\text{res}_{\nu-1}(s'_j - s'_{j-1})^{\nu+1} \rightarrow \text{res}_{\nu-1}(\delta_\nu - \delta_{\nu+1})$ as $j \rightarrow \infty$, we have

$$\|\text{res}_{\nu-1}(\delta_\nu - \delta_{\nu+1}) - \text{res}_{\nu-1}(\beta_{\nu+1} - \beta_\nu)\|_{\nu-1} = 0.$$

By *Proposition 4.1.8*, $\text{res}_{\nu-1}(\beta_{\nu+1} + \delta_{\nu+1}) = \text{res}_{\nu-1}(\beta_\nu + \delta_\nu)$, and this proves (ii).

Let us now verify that $\text{res}_\nu(s'_j)$ is a Cauchy sequence in $\mathcal{S}(S_\nu)$. Indeed, from (48) we see that

$$\|\text{res}_\nu(\beta_{\nu+\mu}) - \text{res}_\nu(\beta_{\nu+\mu-1})\|_\nu \leq \|\text{res}_{\nu+\mu-1}(\beta_{\nu+\mu}) - \text{res}_{\nu+\mu-1}(\beta_{\nu+\mu-1})\|_{\nu+\mu-1} \leq 2^{\nu+\mu-1}.$$

Therefore, for all $i < j$, we simply observe that

$$\|s'_j - s'_i\|_\nu \leq \sum_{\mu=i+1}^j 2^{1-\mu-\nu} \leq 2^{-1-\nu-i},$$

so $s'_j \in \mathcal{S}(S_\nu)$ is a Cauchy sequence, as claimed. \square

The following corollary is immediate.

Corollary 4.2.2. Let $0 \rightarrow \mathcal{R} \rightarrow \mathcal{S} \rightarrow \mathcal{T} \rightarrow 0$ be an exact sequence of coherent analytic sheaves over a Stein space X . Then the sequence $0 \rightarrow \mathcal{R}(X) \rightarrow \mathcal{S}(X) \rightarrow \mathcal{T}(X) \rightarrow 0$ at the level of sections is also exact.

It turns out that the converse of Cartan's Theorem B is also true. That is, X is a Stein space if and only if X is a \mathcal{B} -space. The remainder of this section is devoted to the proof of this fact.

Definition 4.2.3. A map $\sigma : X \rightarrow \mathbb{N}_0$ is called a *cycle* on X if the set-theoretic support $\text{supp}(\sigma) := \{x \in X : \sigma(x) \neq 0\}$ is a discrete set in X .

Definition 4.2.4. Given any cycle σ , there is an associated analytic ideal sheaf $\mathcal{Q}(\sigma)$ which is defined stalkwise by

$$\mathcal{Q}(\sigma) := \bigcup_{x \in X} \mathcal{Q}(\sigma)_x,$$

where $\mathcal{Q}(\sigma)_x := \mathfrak{m}_x^{\sigma(x)}$, and \mathfrak{m}_x is the maximal ideal in $\mathcal{O}_{X,x}$. Note that for any $x \notin \text{supp}(\sigma)$, the stalk of $\mathcal{Q}(\sigma)$ at x is $\mathcal{Q}(\sigma)_x = \mathcal{O}_{X,x}$.

Theorem 4.2.5. The ideal sheaf $\mathcal{Q}(\sigma)$ is coherent.

PROOF. By (ii) of *Proposition 2.3.11*, it suffices to show that $\mathcal{Q}(\sigma)$ is an ideal sheaf of finite type. The identity section $1 \in \mathcal{Q}(\sigma)(X \setminus \text{supp}(\sigma)) = \mathcal{O}(X \setminus \text{supp}(\sigma))$ generates $\mathcal{Q}(\sigma)$ over $X \setminus \text{supp}(\sigma)$. For a point $p \in \text{supp}(\sigma)$, choose a local model (U, \mathcal{O}_U) such that $U \cap \text{supp}(\sigma) = \{p\}$ and $\mathcal{O}_U \cong (\mathcal{O}_G/\mathcal{J})|_U$, where G is a domain in \mathbb{C}^n and \mathcal{J} is an ideal sheaf of finite type. Let $z = (z_1, \dots, z_n)$ denote the coordinates of \mathbb{C}^n , centred at p , and let $J = (j_1, \dots, j_n) \in \mathbb{N}_0^n$ denote a multi-index with $|J| = \sigma(p)$. The monomials $q_J := z^J$ generate the ideal $\mathfrak{m}(\mathcal{O}_{G,p})^{\sigma(p)}$, and therefore the equivalence classes $\bar{q}_J \in \mathcal{O}_G/\mathcal{J}$ generate the ideal $\mathfrak{m}(\mathcal{O}_G/\mathcal{J})_p^{\sigma(p)} =$

$\mathfrak{m}(\mathcal{O}_{U,p})^{\sigma(p)} = \mathfrak{m}_p^{\sigma(p)}$. Since the monomials q_J generate the stalks of \mathcal{O}_G over $G \setminus \{p\}$, the functions $\bar{q}_J|_U \in \mathcal{O}(U)$ generate each stalk $\mathcal{O}_{U,x}$, $x \in U \setminus \{p\}$. Since $U \cap \text{supp}(\sigma) = \{p\}$, $\mathcal{Q}(\sigma)_x = \mathcal{O}_{U,x}$ for all $x \in U \setminus \{p\}$, so $\mathcal{Q}(\sigma)|_U$ is generated by the functions $\bar{q}_J|_U$, and $\mathcal{Q}(\sigma)|_U$ is of finite type. \square

Lemma 4.2.6. Let σ be a cycle on X with $\mathcal{O}_X(X) \rightarrow (\mathcal{O}_X/\mathcal{Q}(\sigma))(X)$ a surjection. Suppose that to every point $p \in \text{supp}(\sigma)$, there is an assigned germ $g_p \in \mathcal{O}_{X,p}$. Then there exists a function $f \in \mathcal{O}_X(X)$ such that

$$f_p - g_p \in \mathfrak{m}_p^{\sigma(p)}, \quad \forall p \in \text{supp}(\sigma).$$

PROOF. Let $\rho : \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{Q}(\sigma)$ denote the quotient map. Observe that since $\text{supp}(\mathcal{O}_X/\mathcal{Q}(\sigma)) = \text{supp}(\sigma)$, define a global section s of $\mathcal{O}_X/\mathcal{Q}(\sigma)$ by

$$s(p) := \begin{cases} g_p, & \forall p \in \text{supp}(\sigma), \\ 0, & \text{otherwise.} \end{cases}$$

Since the associated map $\rho_X : \mathcal{O}_X(X) \rightarrow (\mathcal{O}_X/\mathcal{Q}(\sigma))(X)$ on global sections is surjective, choose $f \in \mathcal{O}_X(X)$ such that $\rho_X(f) = s$. In particular, for each $p \in \text{supp}(\sigma)$, $\rho(f_p) = \rho(g_p)$, i.e., $f_p - g_p \in (\mathcal{Ker}(\rho))_p = \mathcal{Q}(\sigma)_p = \mathfrak{m}_p^{\sigma(p)}$, for all $p \in \text{supp}(\sigma)$. \square

Corollary 4.2.7. Let X be a complex analytic space such that for every coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$, we have $H^1(X, \mathcal{J}) = 0$. Then for any discrete sequence $(x_k)_{k \geq 0}$ in X , and any sequence $(\lambda_k)_{k \geq 0}$ in \mathbb{C} , there exists a function $f \in \mathcal{O}_X(X)$ such that $f(x_k) = \lambda_k$ for all $k \geq 0$.

PROOF. Define a cycle

$$\sigma(x) := \begin{cases} 1, & x = x_k, k \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

i.e., the support of σ is exactly the sequence $(x_k)_{k \geq 0}$, and set $g_{x_k} \in \mathcal{O}_{X,p}$ to be just λ_k , for each $k \geq 0$. By *Lemma 4.2.6*, we can choose a global section $f \in \mathcal{O}_X(X)$ such that $f_{x_k} - g_{x_k} = f_{x_k} - \lambda_k \in \mathfrak{m}_{x_k}$. But this exactly says that for each $k \geq 0$, $f(x_k) = \lambda_k$, as required. \square

Corollary 4.2.8. Let X be a complex analytic space such that $H^1(X, \mathcal{J}) = 0$ for any coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$. Let k denote the embedding dimension of X . Then there are k holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_X(X)$ whose germs $(f_1)_p, \dots, (f_k)_p \in \mathcal{O}_{X,p}$ generate \mathfrak{m}_p as an $\mathcal{O}_{X,p}$ -module.

PROOF. Let $(g_1)_p, \dots, (g_k)_p \in \mathcal{O}_{X,p}$ be the germs which generate \mathfrak{m}_p . Define a cycle $\sigma : X \rightarrow \mathbb{N}_0$ by setting

$$\sigma(x) := \begin{cases} 2, & x = p, \\ 0, & \text{otherwise.} \end{cases}$$

By *Lemma 4.2.6*, there exist $f_j \in \mathcal{O}_X(X)$ such that $(f_j)_p - (g_j)_p \in \mathfrak{m}_p^2$ for each $1 \leq j \leq k$. Hence, the images of each $(f_j)_p$ in $\mathfrak{m}_p/\mathfrak{m}_p^2$ generate the complex vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$. We then simply apply Nakayama's lemma³ to conclude that $(f_1)_p, \dots, (f_k)_p$ generate $\mathcal{O}_{X,p}$ as an $\mathcal{O}_{X,p}$ -module. \square

³Let R be a Noetherian ring with maximal ideal \mathfrak{m} and suppose that M is a finitely generated R -module. Then the elements $x_1, \dots, x_p \in M$ generate the R -module M if and only if their equivalence classes $\bar{x}_1, \dots, \bar{x}_p \in M/\mathfrak{m}M$ generate the R/\mathfrak{m} -vector space $M/\mathfrak{m}M$ (see, e.g., p. 22 of [1, p. 22], [26, p. 72]).

Theorem 4.2.9. Let X be a \mathcal{B} -space, i.e., for any coherent analytic sheaf \mathcal{S} on X , $H^q(X, \mathcal{S}) = 0$ for all $q \geq 1$. Then X is a Stein space.

PROOF. Suppose that X is a \mathcal{B} -space, then for any coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$, it is clear that $H^1(X, \mathcal{J}) = 0$. By *Theorem 4.2.5*, $\mathcal{Q}(\sigma)$ is a coherent ideal sheaf, and so $H^1(X, \mathcal{Q}(\sigma)) = 0$. In particular, the surjectivity assumption of *Lemma 4.2.6* is satisfied. Then from *Corollary 4.2.7*, for any discrete sequence $(x_k)_{k \geq 0}$ in X , we choose $f \in \mathcal{O}_X(X)$ such that $f(x_k) = k$. We claim that X is holomorphically convex. To see this, let $K \subset X$ be a compact set. For any $f \in \mathcal{O}_X(X)$, the continuity of f ensures that $\sup_{z \in K} |\text{red}(f)(z)| < \infty$. By *Lemma 4.2.6* for any infinite discrete sequence $(x_k)_{k \geq 0}$, we may choose $f \in \mathcal{O}_X(X)$ such that $\sup_{k \geq 0} |f(x_k)| = \infty$. The holomorphically convex hull \widehat{K} can therefore not contain any infinite discrete set. To conclude that \widehat{K} is compact, we need only justify that \widehat{K} is closed, but this is (i) of *Lemma 2.1.8*. Let us now show that X is holomorphically separable. Indeed, we simply observe that for distinct $x, y \in X$, *Corollary 4.2.7* permits us to choose $f \in \mathcal{O}_X(X)$ such that $f(x) = 1$ and $f(y) = 0$. Finally, existence of globally-defined coordinate functions is exactly *Corollary 4.2.8*. \square

§4.3. THE COUSIN PROBLEMS

In this section, we discuss the Cousin problems, which were first proposed by P. Cousin in [11]. These problems are the higher-dimensional analogue of the Mittag-Leffler and Weierstrass problems of one complex variable.

We assume throughout this section that $X := (X, \mathcal{O}_X)$ is a reduced complex analytic space.

Definition 4.3.1. The *sheaf \mathcal{K}_X of meromorphic functions* is the sheaf associated to the presheaf of rings of fractions

$$U \longmapsto \mathcal{O}_X(U)[\mathcal{A}(U)]^{-1},$$

where $\mathcal{A}(U)$ denotes the set of all elements $f \in \mathcal{O}_X(U)$ such that $f_x \in \mathcal{O}_{X,x}$ is not a zero-divisor for all $x \in U$. For an open set $U \subseteq X$, elements of $\mathcal{K}_X(U)$ are called *meromorphic functions* (on U).

The First Cousin Problem. Let $\mathfrak{U} = (U_\lambda)_{\lambda \in \Lambda}$ an open cover of X . For each $\lambda \in \Lambda$, let $g_\lambda \in \mathcal{K}_X(U_\lambda)$ be a meromorphic function, and suppose that for each pair $\mu, \lambda \in \Lambda$, the function

$$f_{\lambda\mu} := g_\lambda - g_\mu$$

is holomorphic on $U_\lambda \cap U_\mu$. The *first Cousin problem* asks for the existence of a meromorphic function $f \in \mathcal{K}_X(X)$ such that $f - g_\lambda \in \mathcal{O}_X(U_\lambda)$ for each $\lambda \in \Lambda$. If such a meromorphic function can be found on X , we say that the first Cousin problem is *solvable* on X .

Cartan's theorem B informs us that the first Cousin problem is always solvable on a Stein space.

Theorem 4.3.2. The first Cousin problem is solvable on a Stein space X .

PROOF. The solvability of the first Cousin problem is equivalent to the existence of a surjective map $\mathcal{K}_X(X) \rightarrow (\mathcal{K}_X/\mathcal{O}_X)(X)$. Indeed, if we recall the properties of quotient sheaves, a global section of $\mathcal{K}_X/\mathcal{O}_X$ is specified by an open cover $\mathfrak{U} = (U_\lambda)_{\lambda \in \Lambda}$ of X , and a collection $g_\lambda \in \mathcal{K}_X(U_\lambda)$ such that $g_\lambda - g_\mu \in \mathcal{O}_X(U_\lambda \cap U_\mu)$. The exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\Phi} \mathcal{K}_X \xrightarrow{\Psi} \mathcal{K}_X/\mathcal{O}_X \longrightarrow 0$$

induces a long exact sequence on cohomology,

$$\cdots \longrightarrow H^q(X, \mathcal{O}_X) \xrightarrow{\Phi^*} H^q(X, \mathcal{K}_X) \xrightarrow{\Psi^*} H^q(X, \mathcal{K}_X/\mathcal{O}_X) \xrightarrow{\delta} H^{q+1}(X, \mathcal{O}_X) \longrightarrow \cdots .$$

In particular, we consider the following portion of the above sequence:

$$\cdots \longrightarrow \mathcal{K}_X(X) \xrightarrow{\Psi^*} (\mathcal{K}_X/\mathcal{O}_X)(X) \xrightarrow{\delta} H^1(X, \mathcal{O}_X) \xrightarrow{\Phi^*} H^1(X, \mathcal{K}_X) \longrightarrow \cdots .$$

By *Theorem 4.2.1*, $H^1(X, \mathcal{O}_X) = 0$, and therefore $\Psi^* : \mathcal{K}_X(X) \rightarrow (\mathcal{K}_X/\mathcal{O}_X)(X)$ is surjective, and the first Cousin problem is solvable. \square

Remark 4.3.3. Note that the first Cousin problem is solvable on any space X for which $H^1(X, \mathcal{O}_X) = 0$. In particular, this is true for $\mathbb{C}\mathbb{P}^n$, all projective rational manifolds, and compact Kähler manifolds whose first Betti number is zero (see, e.g., [19, p. 137] and the references therein).

Definition 4.3.4. Let X be a complex analytic space. We denote by \mathcal{O}_X^* the sheaf of invertible elements of \mathcal{O}_X . That is, \mathcal{O}_X^* is the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X^*(U)$, where $\mathcal{O}_X^*(U)$ is the set of invertible elements of $\mathcal{O}_X(U)$. Similarly, we denote by \mathcal{K}_X^* the sheaf of units of \mathcal{K}_X .

The Second Cousin Problem. Let $\mathfrak{U} = (U_\lambda)_{\lambda \in \Lambda}$ an open cover of X . For each $\lambda \in \Lambda$, let $g_\lambda \in \mathcal{K}_X^*(U_\lambda)$ be a meromorphic function. Suppose that for each pair $\mu, \lambda \in \Lambda$, the function

$$f_{\lambda\mu} := \frac{g_\lambda}{g_\mu}$$

is a non-zero holomorphic function on $U_\lambda \cap U_\mu$. The *second Cousin problem* asks for the existence of a meromorphic function $f \in \mathcal{K}_X^*(X)$ such that $\frac{f}{g_\lambda} \in \mathcal{O}_X^*(U_\lambda)$ for each $\lambda \in \Lambda$. If such a meromorphic function can be found on X , we say that the second Cousin problem is *solvable* on X .

Unlike the first Cousin problem, the second Cousin problem is not always solvable on a Stein space. Indeed, there is a topological obstruction measured by the first Chern class. To discuss this topological obstruction, we introduce the following important class of sheaves:

Definition 4.3.5. Let \mathcal{R} be an analytic sheaf on X . We say that \mathcal{R} is a *locally free sheaf* if for any point $x \in X$, there is an open neighbourhood $U \subset X$ of x and a positive integer p , such that $\mathcal{R}|_U \cong \mathcal{O}_X^p|_U$. If this is not only true locally, but globally also, i.e., we may take $U = X$, then \mathcal{R} is said to be a *free sheaf*. The positive integer p such that $\mathcal{R}|_U \cong \mathcal{O}_X^p|_U$ is referred to as the *rank of \mathcal{R} over U* . A locally free sheaf of rank $p = 1$ is called an *invertible sheaf*.

Note that if X is connected, then the rank is the same on each open set, and we, therefore, omit any mention of the open set U .

To elaborate on why the terminology *invertible sheaf* is used, let us note the following elementary properties of invertible sheaves (see, e.g., [28, p. 143]):

- If \mathcal{V} is an invertible sheaf whose dual sheaf is denoted by \mathcal{V}^* , then $\mathcal{V} \otimes \mathcal{V}^* = \mathcal{O}_X$.
- If \mathcal{V}_1 and \mathcal{V}_2 are two invertible sheaves, then $\mathcal{V}_1 \otimes \mathcal{V}_2$ is an invertible sheaf.

Definition 4.3.6. The *Picard group* of X , denoted by $\text{Pic}(X)$, is the set of all (isomorphism classes of) invertible sheaves on X , equipped with a multiplication given by the tensor product.

The Picard group admits the following characterisation in terms of sheaf cohomology:

Theorem 4.3.7. There is an isomorphism

$$\text{Pic}(X) \cong H^1(X, \mathcal{O}_X^*).$$

PROOF. We define a map from $\text{Pic}(X)$ to $H^1(X, \mathcal{O}_X^*)$ as follows. Choose an open cover $\mathfrak{U} := (U_\lambda)_{\lambda \in \Lambda}$ of X , and let $\text{Pic}(\mathfrak{U})$ be the set of all isomorphism classes of invertible sheaves which are trivialised over each U_λ . That is, if $\mathcal{V} \in \text{Pic}(\mathfrak{U})$, for each $\lambda \in \Lambda$, there are isomorphisms $\varphi_\lambda : \mathcal{O}_{U_\lambda} \rightarrow \mathcal{V}|_{U_\lambda}$, and for any pair $\lambda, \mu \in \Lambda$, we may define transition maps $g_{\lambda\mu} : \mathcal{O}_{U_\lambda \cap U_\mu} \rightarrow \mathcal{O}_{U_\lambda \cap U_\mu}$, $g_{\lambda\mu} := \varphi_\mu \circ \varphi_\lambda^{-1} \in \mathcal{O}_X^*(U_\lambda \cap U_\mu)$. It is easy to verify that

$$\begin{cases} g_{\lambda\mu} \circ g_{\mu\lambda} & = 1, \\ g_{\lambda\mu} \circ g_{\mu\eta} \circ g_{\eta\lambda} & = 1, \end{cases}$$

i.e., $(g_{\lambda\mu})$ defines a Čech 1-cocycle. The map $\text{Pic}(\mathfrak{U}) \rightarrow \check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$ is then defined by $\mathcal{V} \mapsto (g_{\lambda\mu})$.

Let us show that this map does not depend on the choice of isomorphisms $(\varphi_\lambda)_{\lambda \in \Lambda}$. Indeed, let $(\psi_\lambda)_{\lambda \in \Lambda}$ denote another collection of isomorphisms $\psi_\lambda : \mathcal{O}_{U_\lambda} \rightarrow \mathcal{V}|_{U_\lambda}$. The ratio $\psi_\lambda / \varphi_\lambda$ is given by some $h_\lambda \in \mathcal{O}_X^*(U_\lambda)$, and therefore, the transition maps defined by $(\psi_\lambda)_{\lambda \in \Lambda}$ are given by

$$\tilde{g}_{\lambda\mu} := \frac{\psi_\lambda}{\psi_\mu} = \frac{\varphi_\lambda}{\varphi_\mu} \cdot \frac{h_\lambda}{h_\mu} = g_{\lambda\mu} \cdot \frac{h_\lambda}{h_\mu}.$$

Since $h_\lambda \circ h_\mu^{-1}$ is a Čech coboundary however, $\tilde{g}_{\lambda\mu}$ and $g_{\lambda\mu}$ define the same element of $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$.

Now suppose that we have another open cover \mathfrak{V} over X that trivialises \mathcal{V} . By the above argument, we can define a map $\text{Pic}(\mathfrak{V}) \rightarrow \check{H}^1(\mathfrak{V}, \mathcal{O}_X^*)$. Let \mathfrak{D} denote the common refinement of \mathfrak{U} and \mathfrak{V} . The trivialisation of \mathcal{V} over \mathfrak{U} induces a trivialisation of \mathcal{V} over \mathfrak{D} , given simply by restriction. Similarly, the trivialisation of \mathcal{V} over \mathfrak{V} induces a trivialisation of \mathcal{V} over \mathfrak{D} , again by restriction. We then obtain homomorphisms $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*) \rightarrow \check{H}^1(\mathfrak{D}, \mathcal{O}_X^*)$ and $\check{H}^1(\mathfrak{V}, \mathcal{O}_X^*) \rightarrow \check{H}^1(\mathfrak{D}, \mathcal{O}_X^*)$ which the diagram

$$\begin{array}{ccc}
 \check{H}^1(\mathfrak{U}, \mathcal{O}_X^*) & \xrightarrow{\quad} & \check{H}^1(\mathfrak{V}, \mathcal{O}_X^*) \\
 & \searrow \quad \swarrow & \\
 & \check{H}^1(\mathfrak{D}, \mathcal{O}_X^*) &
 \end{array}$$

commutative. Hence, since we have a map $\text{Pic}(\mathfrak{U}) \rightarrow \check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$ for each open cover, we may take an inductive limit, and this yields a map $\text{Pic}(X) \rightarrow H^1(X, \mathcal{O}_X^*)$.

Now, to define a map $H^1(X, \mathcal{O}_X^*) \rightarrow \text{Pic}(X)$. Let $\alpha \in \check{H}^1(X, \mathcal{O}_X^*)$, we observe that since $H^1(X, \mathcal{O}_X^*)$ is an inductive limit of $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$ over all open covers \mathfrak{U} , we may choose an open cover \mathfrak{U} and a Čech 1-coycle $(g_{\lambda\mu})$ in $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$ that represents the isomorphism class α . We then construct an invertible sheaf \mathcal{V} in the obvious way whose transition maps are $(g_{\lambda\mu})$ with respect to the open cover \mathfrak{U} . \square

Definition 4.3.8. A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$, where \mathcal{K}_X^* denotes the sheaf of units of \mathcal{K}_X .

Note that a Cartier divisor on X is specified by an open cover $(U_\lambda)_{\lambda \in \Lambda}$ of X , and a collection $g_\lambda \in \mathcal{K}_X^*(U_\lambda)$ such that $\frac{g_\lambda}{g_\mu} \in \mathcal{O}_X^*(U_\lambda \cap U_\mu)$.

Proposition 4.3.9. The second Cousin problem is solvable on X if the Picard group of X is trivial, i.e., $\text{Pic}(X) = 0$.

PROOF. In a similar manner to the proof of *Theorem 4.3.2*, we need to show that there is a surjective map $\psi^* : \mathcal{K}_X^*(X) \rightarrow (\mathcal{K}_X^*/\mathcal{O}_X^*)(X)$. To this end, if $\text{Pic}(X) = 0$, then by *Theorem 4.3.7* $H^1(X, \mathcal{O}_X^*) = 0$. The exact sequence

$$1 \longrightarrow \mathcal{O}_X^* \xrightarrow{\varphi} \mathcal{K}_X^* \xrightarrow{\psi} \mathcal{K}_X^*/\mathcal{O}_X^* \longrightarrow 0$$

induces an exact sequence on cohomology

$$1 \longrightarrow \mathcal{O}_X^*(X) \xrightarrow{\varphi^*} \mathcal{K}_X^*(X) \xrightarrow{\psi^*} (\mathcal{K}_X^*/\mathcal{O}_X^*)(X) \xrightarrow{\delta_1} H^1(X, \mathcal{O}_X^*) \longrightarrow \dots$$

Since $H^1(X, \mathcal{O}_X^*) = 0$, ψ^* is a surjection, and the second Cousin problem is solvable. \square

Lemma 4.3.10. The *exponential sequence* of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1$$

is exact.

PROOF. The map $\exp : \mathcal{O}_X \rightarrow \mathcal{O}_X^*$ is surjective. Indeed, it suffices to show that this is true on stalks. To this end, fix a point $x \in X$, and let $f \in \mathcal{O}_{X,x}^*$. Choosing a branch of the logarithm in a sufficiently small neighbourhood of x , yields an analytic function $\log(f) \in \mathcal{O}_{X,x}$. It is clear that $\exp(\log(f)) = f$ in this neighbourhood of x . Let $\mathcal{K}\text{er}(\exp)$ denote the kernel sheaf of the exponential map. We claim that $\mathcal{K}\text{er}(\exp)$ is the constant sheaf $2\pi i\mathbb{Z}$. It is immediate that $2\pi i\mathbb{Z}$ is a subsheaf of $\mathcal{K}\text{er}(\exp)$. We therefore concern ourselves only with showing that $\mathcal{K}\text{er}(\exp) \subset 2\pi i\mathbb{Z}$. As before, fix a point $x \in X$, and let $f \in \mathcal{O}_{X,x}$ such that $\exp(f) = 1$. Multiplying f by an appropriate integral multiple of $2\pi i$, we may assume that $f \in \mathfrak{m}_x$, i.e., we

may assume that $f(x) = 0$. The power series expansion of $\exp(z)$ yields an analytic function $g \in \mathcal{O}_{X,x}$ such that $\exp(f) = 1 + f - g \cdot f^2$. Since $\exp(f) = 1$, proceeding inductively, we observe that

$$f = g \cdot f^2 = \dots = g^k \cdot f^{k+1} \in \bigcap_{k=1}^{\infty} \mathfrak{m}_x^k = \{0\}, \quad (51)$$

where the last equality follows from (ii) of (1). Hence, we see that $f = 0$, and therefore $\mathcal{K}er(\exp) = 2\pi i\mathbb{Z}$, as required. \square

We now have two exact sequences:

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow (\mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1.$$

These sequences induce the following exact sequences of sheaf cohomology groups:

$$\dots \rightarrow H^q(X, \mathcal{O}_X^*) \rightarrow H^q(X, \mathcal{K}_X^*) \rightarrow H^q(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \xrightarrow{\delta_q} H^{q+1}(X, \mathcal{O}_X^*) \rightarrow \dots$$

and

$$\dots \rightarrow H^{q+1}(X, \mathbb{Z}) \rightarrow H^{q+1}(X, \mathcal{O}_X) \rightarrow H^{q+1}(X, \mathcal{O}_X^*) \xrightarrow{\tilde{\delta}_{q+1}} H^{q+2}(X, \mathbb{Z}) \rightarrow \dots$$

respectively. By combining the connecting homomorphisms δ_q and $\tilde{\delta}_{q+1}$, we obtain a map

$$c_{q+1} : H^q(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow H^{q+2}(X, \mathbb{Z}); \quad D \mapsto \tilde{\delta}_{q+1} \circ \delta_q(D). \quad (52)$$

Definition 4.3.11. If $q = 1$ in (52), we call the image of Cartier divisor D , under the map $c_1 : (\mathcal{K}_X^*/\mathcal{O}_X^*)(X) \rightarrow H^2(X, \mathbb{Z})$, the *first Chern class of D* .

Theorem 4.3.12. Let X be a complex analytic space with $H^1(X, \mathcal{O}_X) = 0$. The second Cousin problem is solvable on X if and only if $c_1(D) = 0$ for all Cartier divisors D .

PROOF. For the second Cousin problem to be solvable, we require that the map

$$\psi : \mathcal{K}_X^*(X) \longrightarrow (\mathcal{K}_X^*/\mathcal{O}_X^*)(X)$$

is surjective. We have the exact sequence

$$\mathcal{K}_X^*(X) \xrightarrow{\psi^*} (\mathcal{K}_X^*/\mathcal{O}_X^*)(X) \xrightarrow{\delta_1} H^1(X, \mathcal{O}_X^*).$$

Hence, for a Cartier divisor D , $D \in \text{Im}(\psi^*)$ if and only if $\delta_1(D) = 0$. In particular, this implies that $\tilde{\delta}_2 \circ \delta_1(D) = 0$. If $H^1(X, \mathcal{O}_X) = 0$, then $\tilde{\delta}_2$ is injective, which proves the converse. \square

Remark 4.3.13. It is not, in general, true that the second Cousin problem is solvable if X is Stein. For example, by (ii) of *Lemma 2.1.8*, the complex manifold $\mathbb{C}^\times \times \mathbb{C}^* \subset \mathbb{C}^2$ is clearly Stein. But $\mathbb{C}^* \times \mathbb{C}^*$ deformation retracts onto $\mathbb{S}^1 \times \mathbb{S}^1$, so $H^2(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z}) = H^2(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z}) = \mathbb{Z}$. By *Theorem 4.3.12*, the second Cousin problem is not solvable on $\mathbb{C}^* \times \mathbb{C}^*$.

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