Curvature Aspects of Hyperbolicity and Non-Hyperbolicity in Complex Geometry

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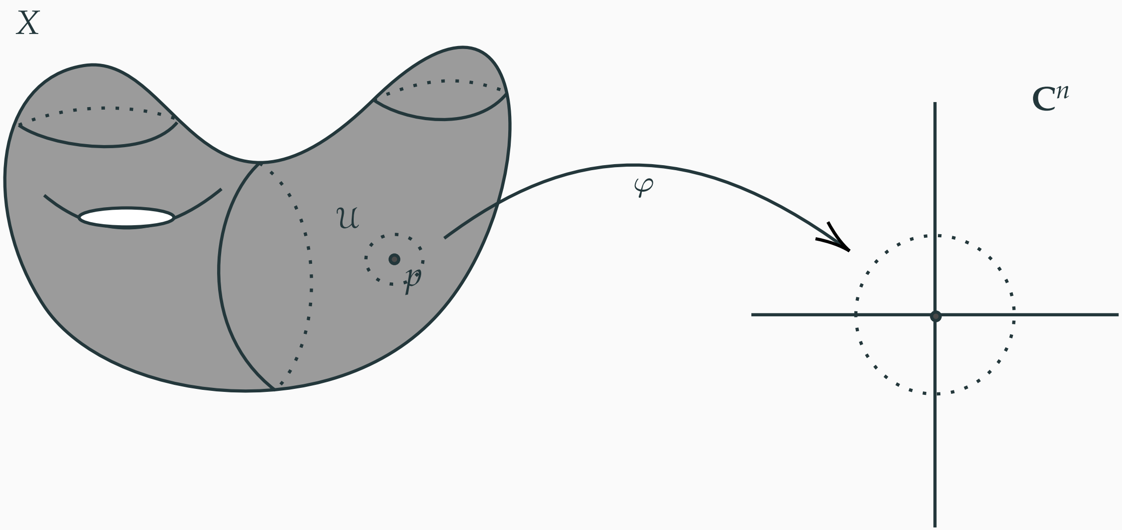
Complex Manifolds

A complex manifold X is a space that is locally modelled on $\mathbf{C}^n$ in such a way that the local analytic structure is preserved.



Examples: Euclidean space $\mathbf{C}^n$; projective space $\mathbf{P}^n$; the ball $\mathbf{B}^n$; Tori $\mathbf{T}^n$; Calabi–Eckman manifolds $\mathbf{S}^{2k+1} \times \mathbf{S}^{2\ell+1}$; the spheres $\mathbf{S}^2$ and $\mathbf{S}^6$ (?);

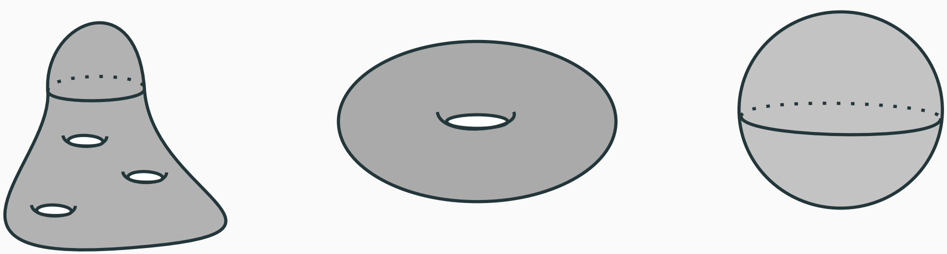
The Uniformization Theorem

Model geometries.

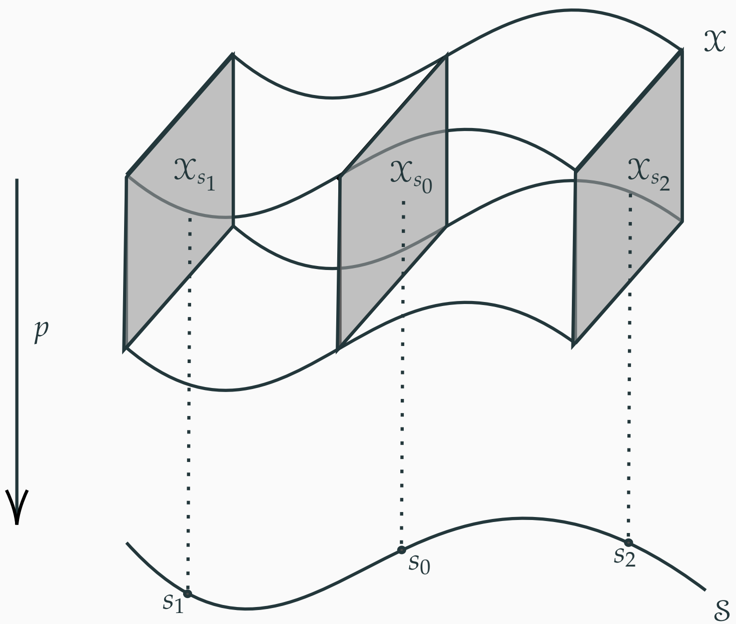
Theorem. A compact Riemann surface X of genus $g := \frac{1}{2}b_1(X)$ has a metric with

- positive curvature $K > 0 \iff g = 0 \iff X \simeq \mathbf{P}^1$;
- vanishing curvature $K = 0 \iff g = 1 \iff X \simeq \mathbf{C}/\Lambda$;
- negative curvature $K < 0 \iff g \geq 2 \iff X \simeq \mathbf{D}/\Gamma$.

$K < 0$ $(g \geq 2)$ | $K = 0$ $(g = 1)$ | $K > 0$ $(g = 0)$



Definition. A family of complex manifolds (or a holomorphic fiber space) is a surjective holomorphic map $p : \mathcal{X} \to \mathcal{S}$ between complex manifolds with connected fibers $\mathcal{X}_s := p^{-1}(s)$.



Trivial Fiber Spaces

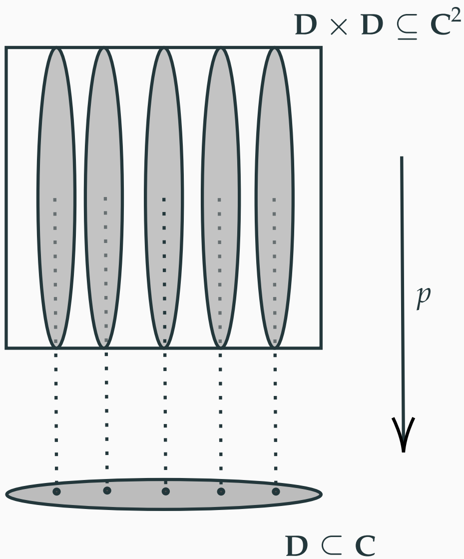
The simplest examples of fiber spaces are products $p : \mathcal{X} \times \mathcal{S} \to \mathcal{S}$, with $p(x, s) = s$. Slightly less-trivial are the fiber bundles:

<u>Theorem.</u> (Fischer–Grauert). A holomorphic fiber space $p : \mathcal{X} \to \mathcal{S}$ with compact fibers $\mathcal{X}_s$ is a fiber bundle if and only if all fibers are biholomorphic.

A family/fiber space with compact fibers will be said to have (non-trivial) holomorphic variation if the fibers are not all biholomorphic.

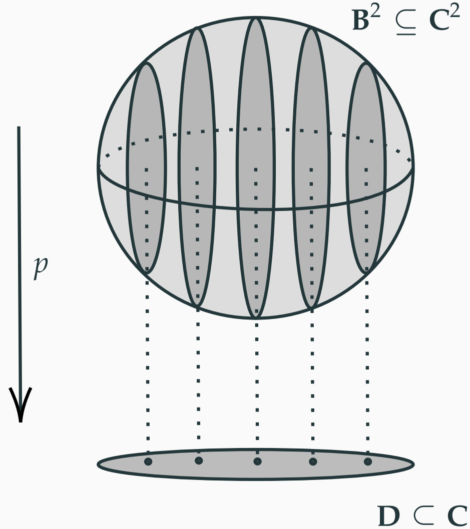
Families of Complex Manifolds

Let $\mathbf{D} \times \mathbf{D} := \{(z, w) \in \mathbf{C}^2 : |z| < 1, |w| < 1\}$ denote the bidisk in $\mathbf{C}^2$. Let $p : \mathbf{D} \times \mathbf{D} \to \mathbf{D}$, with $p(z, w) := w$. Then p defines a holomorphic fiber space, with each fiber $p^{-1}(t)$ (for $t \in \mathbf{D}$) biholomorphic to the unit disk $\mathbf{D}$.



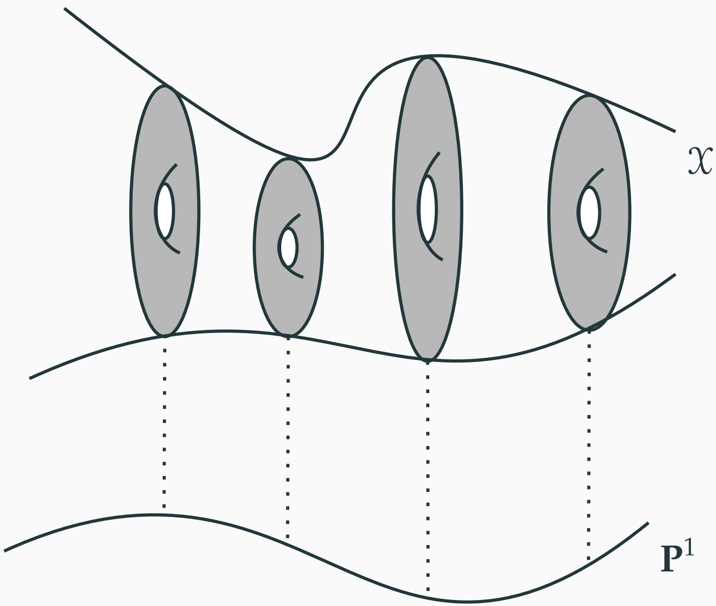
Families of Complex Manifolds

Let $\mathbf{B}^2 := \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 < 1\}$ denote the unit ball in $\mathbf{C}^2$. Denote by $p : \mathbf{B}^2 \to \mathbf{D}$, with $p(z, w) := w$. Then p defines a holomorphic fiber space, with each fiber $p^{-1}(t)$ (for $t \in \mathbf{D}$) biholomorphic to the unit disk $\mathbf{D}$.

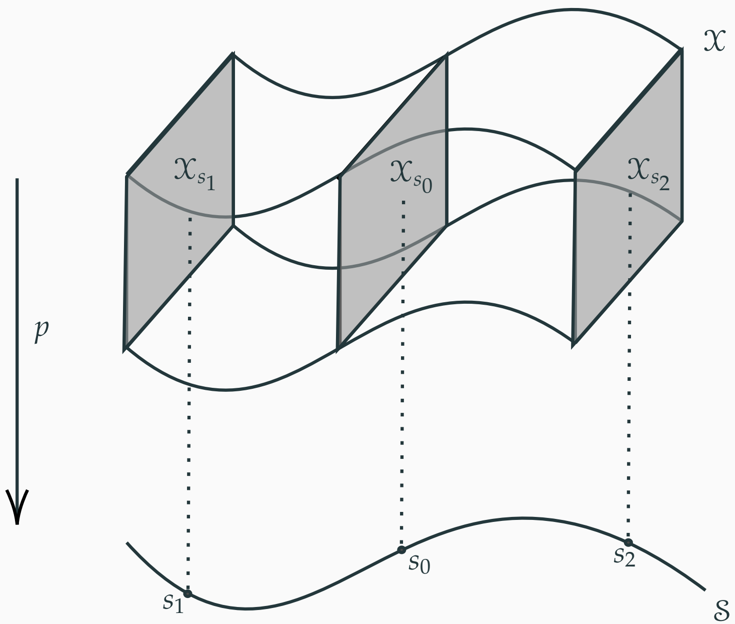


An Example with Compact Fibers

Elliptic K3 surfaces: the total space $\mathcal{X}$ of a holomorphic fiber space $p : \mathcal{X} \to \mathbf{P}^1$, with every smooth fiber being an elliptic curve.



In the study of families $p : \mathcal{X} \to \mathcal{S}$ of complex manifolds, there are four key aspects. The properties of (1) the total space $\mathcal{X}$; (2) the base space $\mathcal{S}$; (3) the fibers $\mathcal{X}_s = p^{-1}(s)$; (4) the (holomorphic) variation in the fibers.



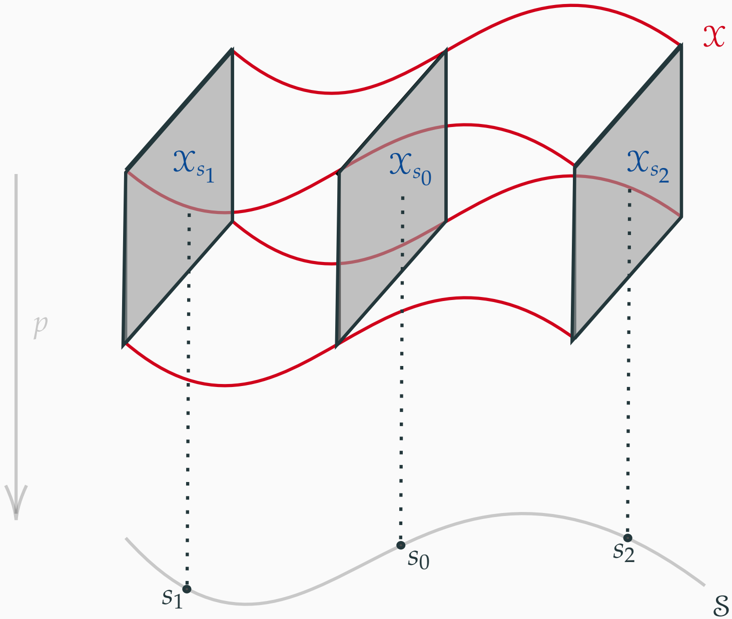
$\mathcal{X} =$ total space

$\mathcal{S} =$ base space

$\mathcal{X}_s =$ fibers

The interaction between the total space $\mathcal{X}$ and the fibers $\mathcal{X}_s$ is well known:

The curvature of the fibers $\mathcal{X}_s$ is bounded from above by the curvature of $\mathcal{X}$.



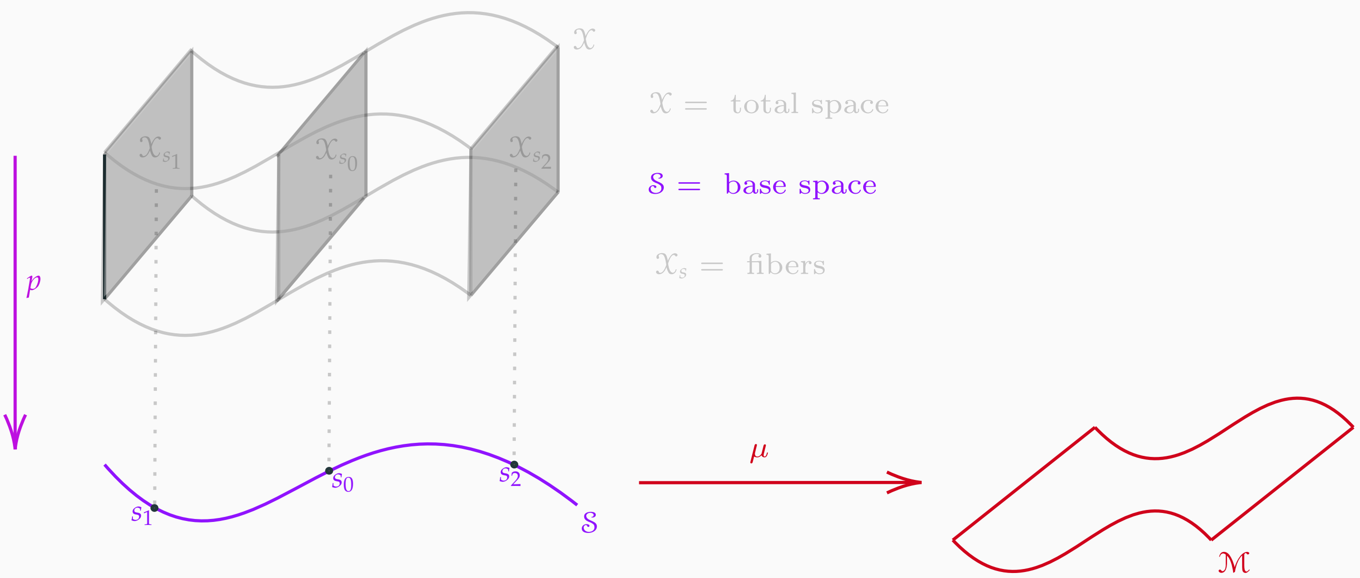
$\mathcal{X} =$ total space

$\mathcal{S} =$ base space

$\mathcal{X}_s =$ fibers

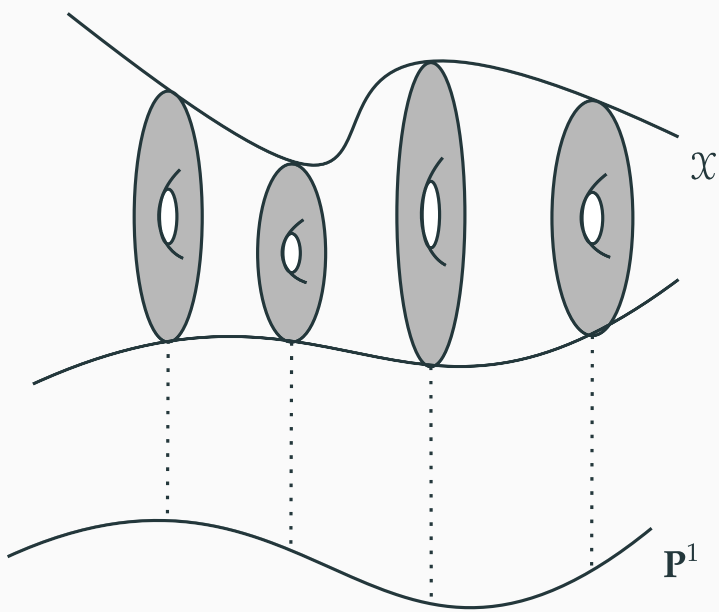
$$\mathrm{Curv}(\mathcal{X}_s) \leq \mathrm{Curv}(\mathcal{X})$$

The interaction between the base space $\mathcal{S}$ and the holomorphic variation has received considerable attention since the holomorphic variation in the fibers of many classes of (compact) complex manifolds is encoded in the moduli map $\mu : \mathcal{S} \to \mathcal{M}$.



There are also some intimations on the relationship between the base $\mathcal{S}$, the fibers $\mathcal{X}_s$ and the holomorphic variation in the fibers.

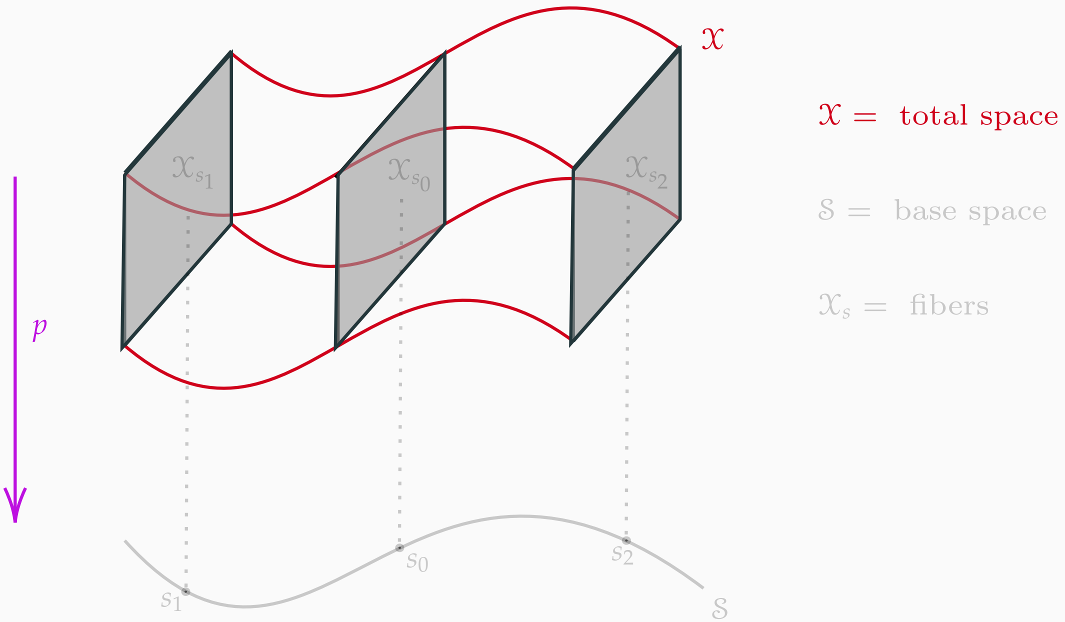
Example (elliptic K3 surfaces): Let $\mathcal{X}$ be the total space of a holomorphic fiber space $p : \mathcal{X} \to \mathbf{P}^1$ with the fibers being elliptic curves.



Example (elliptic K3 surfaces): Let $\mathcal{X}$ be the total space of a holomorphic fiber space $p : \mathcal{X} \to \mathbf{P}^1$ with the fibers being elliptic curves.

The holomorphic structure of an elliptic curve is parametrized by the j–invariant. If all fibers of an elliptic K3 are smooth, we get a holomorphic function $j : \mathbf{P}^1 \to \mathbf{C}$ which is constant by the maximum principle.

We understand very little, however, about the interaction between the total space $\mathcal{X}$ and the holomorphic variation in the fibers.



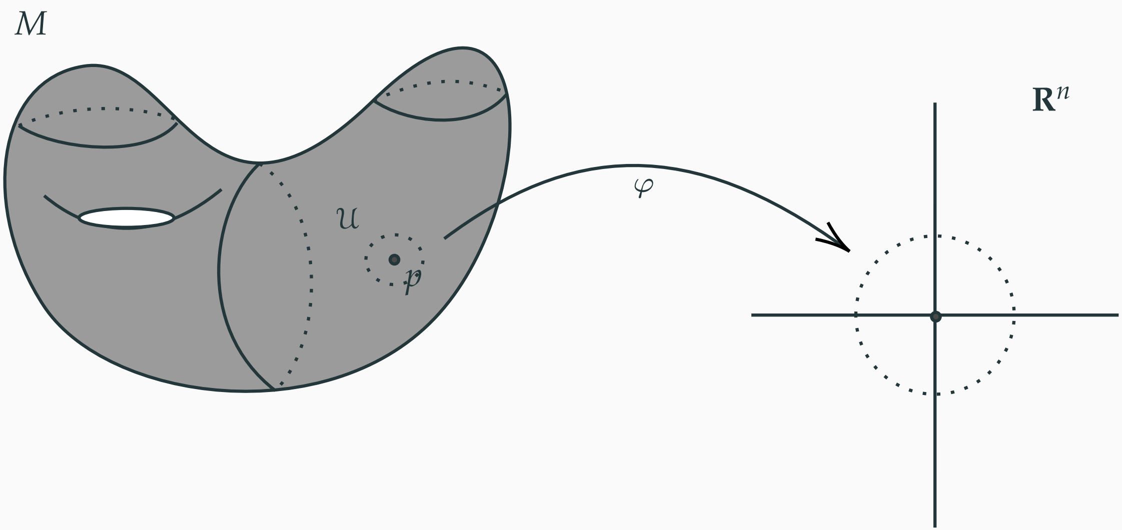
<u>Question</u>. Let $p : \mathcal{X} \to \mathcal{S}$ be a holomorphic family of complex manifolds. How does the curvature of $\mathcal{X}$ influence/interact with the holomorphic variation of p?

Recently, a picture has emerged on the relationship that the curvature of $\mathcal{X}$ has on the holomorphic variations of the fibers.

Surprisingly, the picture emerges from Riemannian geometry, with no reference to the holomorphic structure.

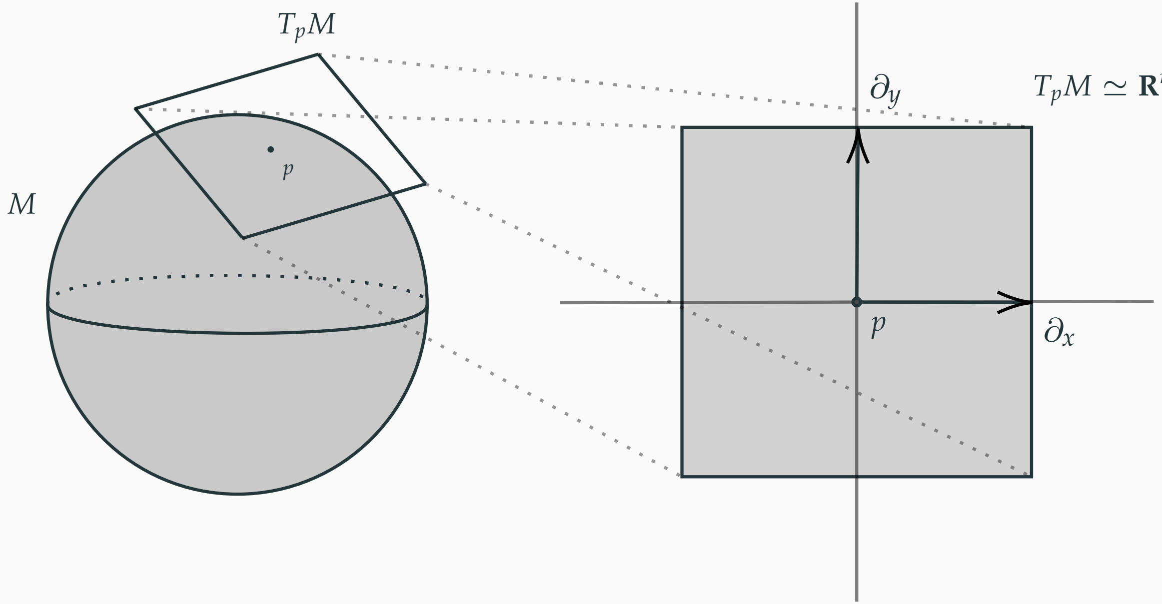
The Tangent Space

Let M be a smooth manifold. For any point $p \in M$, there is an open neighborhood $\mathcal{U} \subset M$ containing p and a homeomorphism $\varphi : \mathcal{U} \to \mathbf{B}^n \subseteq \mathbf{R}^n$ mapping p to the origin in $\mathbf{R}^n$. If $(x_1, ..., x_n)$ denote the coordinates on $\mathbf{R}^n$, we can pull them back via φ to provide M with coordinates.



The Tangent Space T_pM

From these local coordinates, we can define coordinate partial derivatives $\partial_{x_k} := \frac{\partial}{\partial x_k}$, that we can view as vectors tangent to M at the point $p \in M$. The $\mathbf{R}$–linear span of the set $\{\partial_{x_1}, ..., \partial_{x_n}\}$ forms an n–dimensional vector space T_pM – the tangent space to M at the point p.



Riemannian Metrics

Let M be a smooth manifold as before, with tangent space T_pM. Let $g_p : T_pM \times T_pM \to \mathbf{R}$ be a positive-definite quadratic form on T_pM.

Definition. A Riemannian metric g on M is smooth assignment of positive-definite quadratic forms g_p on each tangent space T_pM.

The Riemannian metric allows us to compute the lengths of tangent vectors, and by integrating, the lengths of curves in the manifold.

Definition. Let (M, g) be a Riemannian manifold. If $\gamma : [0,1] \to M$ is a smooth curve, then the length of γ is defined by

$$\text{length}_g(\gamma) \; := \; \int_0^1 |\dot{\gamma}(t)|^2_{g_{\gamma(t)}} dt.$$

This, in turn provides a notion of distance:

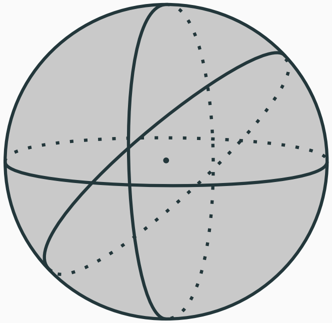
Definition. Let (M, g) be a Riemannian manifold. The distance between two points $p, q \in M$ is defined by

$$\text{dist}_g(p, q) \; := \; \inf_{\gamma} \text{length}_g(\gamma),$$

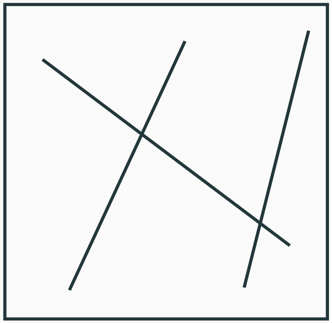
where the infimum is over all smooth curves $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Geodesics

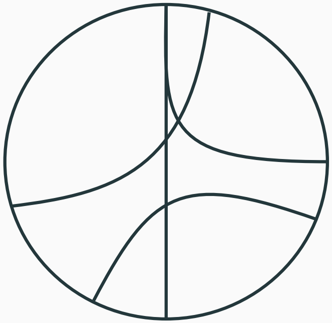
A geodesic is a curve in M which locally minimizes the distance between any two points.



Geodesics on $\mathbf{P}^1$



Geodesics on $\mathbf{C}$

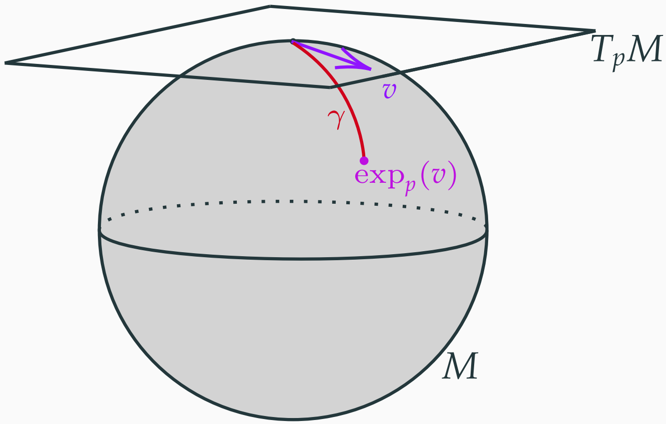


Geodesics on $\mathbf{D}$

The exponential map

$$\exp_p : T_pM \to M, \qquad \exp_p(v) := \gamma(1),$$

where $\gamma : [0,1] \to M$ is unique geodesic satisfying $\gamma(0) = p$ and $\gamma'(0) = v$.



The exponential map provides a canonical set of coordinates on a Riemannian manifold.

The Curvature Tensor

The Taylor expansion of the components of the Riemannian metric g in the exponential coordinates $(x_1, ..., x_n)$:

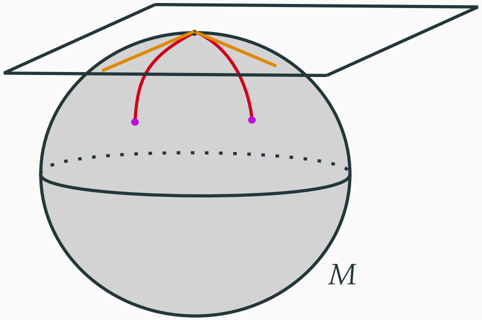
$$g(\partial_{x_i}, \partial_{x_j}) \quad = \quad \delta(\partial_{x_i}, \partial_{x_j}) - \frac{1}{3} R_{ikj\ell} x^k x^\ell + O(|x|^3).$$

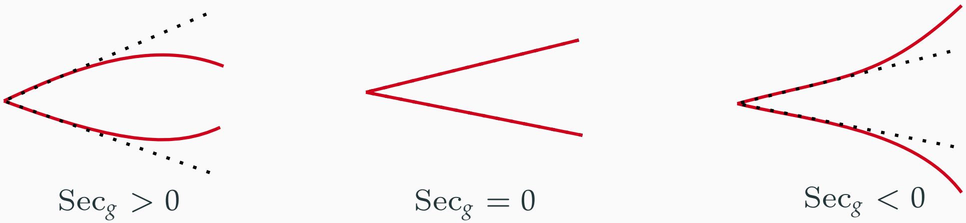
The Riemannian curvature tensor measures the failure of the exponential map to be an isometry.

The Sectional Curvature

From the Riemannian curvature tensor, we can define the sectional curvature:

$$\mathrm{Sec}_g(u, v) \;:=\; \frac{R(u, v, v, u)}{|u|^2|v|^2 - g(u, v)^2}$$

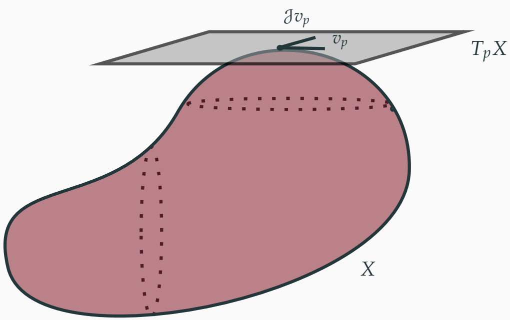


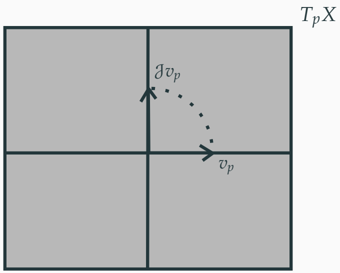


Theorem. (Priessmann). Let (M, g) be a compact Riemannian manifold with $\mathrm{Sec}_g < 0$. Then M is not homeomorphic to a product.

Reminder: Complex Structures

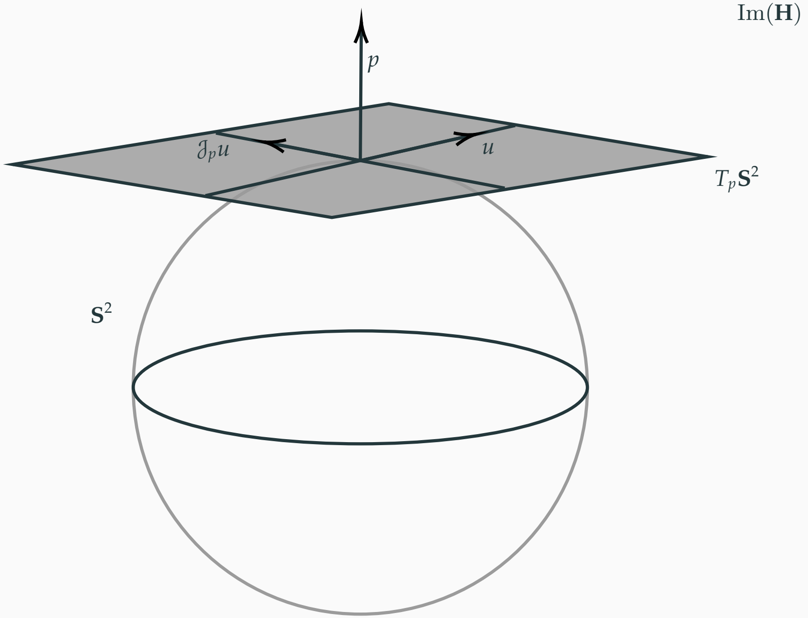
The complex structure of a complex manifold X can be encoded in an endomorphism $\mathcal{J} : TX \to TX$ satisfying $\mathcal{J}^2 = -\mathrm{id}$ together with an integrability crtierion.





A Complex Structure on $\mathbf{S}^2$

Identify $\mathbf{S}^2 \subset \mathbf{R}^3$ with the space of unit imaginary quaternions $\mathrm{Im}(\mathbf{H}^3) \simeq \mathbf{R}^3$. For each point $p \in \mathbf{S}^2$, we get a map $\mathcal{J}_p : T_p\mathbf{S}^2 \to T_p\mathbf{S}^2$ satisfying $\mathcal{J}_p^2 = -\mathrm{id}_{T_p\mathbf{S}^2}$, given by $\mathcal{J}_p(v) := p \times v$.



For this complex structure, $\mathcal{J}$ is integrable $\iff$ the multiplication on $\mathbf{H}$ is associative.

Hermitian and Kähler Metrics

Definition. A Riemannian metric g is said to be Hermitian if

$$g(\mathcal{J}\cdot, \mathcal{J}\cdot) \;=\; g(\cdot, \cdot).$$

If the 2–form $\omega_g(\cdot, \cdot) := g(\mathcal{J}\cdot, \cdot)$ is closed, g is said to be Kähler.

Kähler Examples: $\mathbf{C}^n$; $\mathbf{B}^n$; $\mathbf{P}^n$; submanifolds (hence, projective and Stein manifolds are Kähler).

Non-Kähler Examples: $\mathbf{S}^1 \times \mathbf{S}^3$; if $\mathbf{S}^6$ has an integrable complex structure, it will not be Kähler; the flag manifold $F_{1,2,3}(\mathbf{C}^3) := \mathrm{SU}(3)/S(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1))$ is projective (hence, Kähler), but the Killing metric on $F_{1,2,3}(\mathbf{C}^3)$ is not Kähler;

The Holomorphic Bisectional Curvature

We want to understand Priessmann's theorem in the Hermitian category. Recall:

Theorem. (Priessmann). Let (M, g) be a compact Riemannian manifold with $\text{Sec}_g < 0$. Then M is not homeomorphic to a product.

For Hermitian metrics ω, the most natural replacement for the sectional curvature is the Holomorphic Bisectional Curvature

$$\text{HBC}_\omega(u, v) \ := \ \frac{1}{|u|_\omega^2 |v|_\omega^2} R(u, \overline{u}, v, \overline{v}).$$

If the metric ω is Kähler, then the holomorphic bisectional curvature is a sum of two sectional curvatures.

Holomorphic Bisectional Curvature Examples

For Hermitian metrics ω, the most natural replacement for the sectional curvature is the Holomorphic Bisectional Curvature

$$\mathrm{HBC}_\omega(u, v) \;:=\; \frac{1}{|u|_\omega^2 |v|_\omega^2} R(u, \overline{u}, v, \overline{v}).$$

Examples: Compact Hermitian with quasi-positive HBC is biholomorphic to $\mathbf{P}^n$ (Mori, Siu–Yau, Ustinovskiy); the Bergman metric on $\mathbf{B}^n$ has $\mathrm{HBC}_\omega \leq -1$; there are compact simply connected manifolds with $\mathrm{HBC}_\omega < 0$ (Mohsen).

A Theorem of Paul Yang

Theorem. (Priessmann). Let (M, g) be a compact Riemannian manifold with $\mathrm{Sec}_g < 0$. Then M is not homeomorphic to a product.

The first extension of this to the complex-analytic setting is due to Yang:

Theorem. (P. Yang). Let $p : \mathcal{X} \to \mathcal{S}$ be a holomorphic fiber space with compact Kähler fibers. If all fibers of p are biholomorphic, then $\mathcal{X}$ does not admit a metric with $\mathrm{HBC}_\omega < 0$.

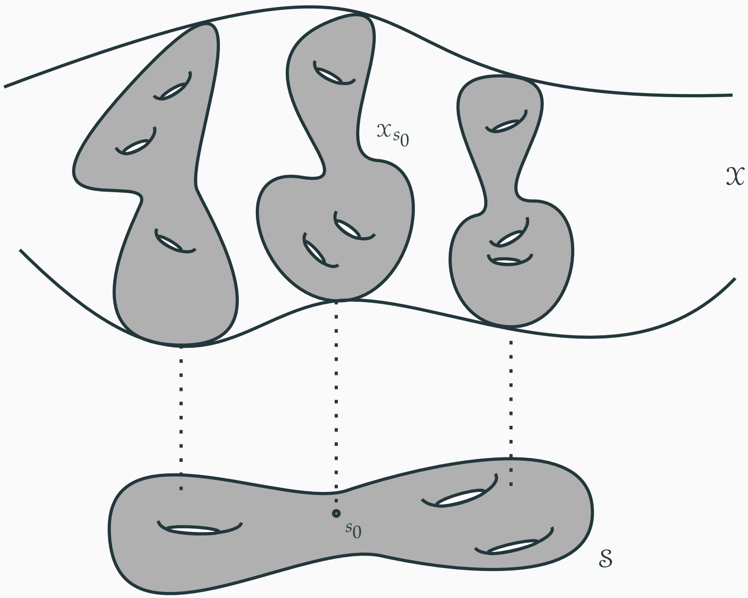
There have been a number of extensions of Yang's theorem (H. Seshadri, F. Zheng, V. Tosatti, K. Tang) all viewing Yang's theorem as an extension of Priessmann's theorem.

A Question Raised By Mok

Still out of reach, however, is the following long-standing question raised by Ngaiming Mok:

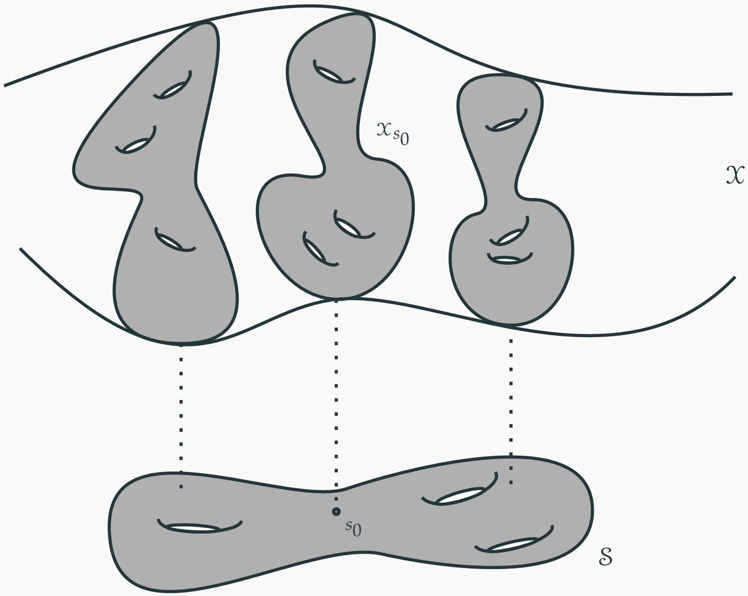
Question. Does the bidisk $\mathbf{D} \times \mathbf{D}$ admit a complete Kähler metric with $\mathrm{HBC}_\omega \leq -1$?

Definition. A Kodaira fibration surface $\mathcal{X}$ is the total space of a non-trivial holomorphic fiber space $p : \mathcal{X} \to \mathcal{S}$, where the base and fibers are compact Riemann surfaces of genus ≥ 2.



A Theorem of To and Yeung

Theorem. (To–Yeung). The total space $\mathcal{X}$ of a Kodaira fibration surface admits a Kähler metric with $\mathrm{HBC}_\omega < 0$.



We saw before:

<u>Theorem</u>. (P. Yang). Let $p : \mathcal{X} \to \mathcal{S}$ be a trivial Kodaira fibration surface. Then $\mathcal{X}$ does not admit a metric with $\mathrm{HBC}_{\omega} < 0$.

On the other hand:

<u>Theorem</u>. (To–Yeung). The total space $\mathcal{X}$ of a Kodaira fibration surface admits a Kähler metric with $\mathrm{HBC}_{\omega} < 0$.

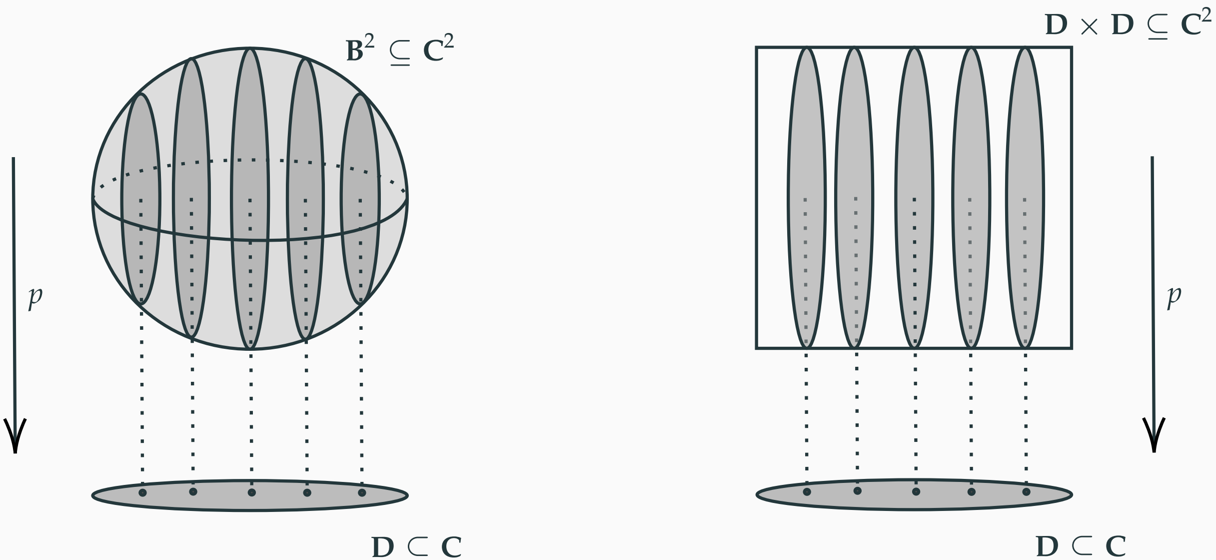
Taken together, the theorems of To–Yeung and Yang illuminate the following:

<u>Theorem</u>. Let $p : \mathcal{X} \to \mathcal{S}$ be a Kodaira fibration surface. Then p has non-trivial variation if and only if $\mathcal{X}$ admits a Kähler metric with $\mathrm{HBC}_{\omega} < 0$.

Simplest Case of Non-Compact Fiber Spaces

Definition. A surjective holomorphic map $p : \mathcal{X} \to \mathbf{D}$ is called a disk fibration if every fiber $\mathcal{X}_s := p^{-1}(s)$ is biholomorphic to the unit disk $\mathbf{D} \subset \mathbf{C}$.

We saw the following examples before:



Holomorphic Variation in Disk Fibrations

For compact fiber spaces, $p : \mathcal{X} \to \mathcal{S}$ is locally trivial $\iff$ the fibers are all biholomorphic.

Theorem. (Royden). A disk fibration $p : \mathcal{X} \to \mathbf{D}$ is locally trivial if and only if $\mathcal{X} \simeq \mathbf{D} \times \mathbf{D}$ and $p : \mathbf{D} \times \mathbf{D} \to \mathbf{D}$, with $p(z, w) = w$.

Example. The disk fibration $p : \mathbf{B}^2 \to \mathbf{D}$ given by projection onto one of the factors is not locally trivial. The Bergman metric has $\mathrm{HBC}_\omega \leq -\kappa_0 < 0$.

Returning to the Mok problem:

Question. Does the bidisk $\mathbf{D} \times \mathbf{D}$ admit a complete Kähler metric with $\mathrm{HBC}_\omega \leq -1$?

Comparing with the case of Kodaira fibration surfaces:

Theorem. Let $p : \mathcal{X} \to \mathcal{S}$ be a Kodaira fibration surface. Then p has non-trivial variation if and only if $\mathcal{X}$ admits a Kähler metric with $\mathrm{HBC}_\omega < 0$.

A New Perspective on the Mok Problem

A resolution of the Mok problem would be achieved by proving the following more general statement:

Conjecture. Let $p : \mathcal{X} \to \mathbf{D}$ be a disk fibration. If $\mathcal{X}$ admits a Kähler metric with $\mathrm{HBC}_\omega \leq -1$, then p has non-trivial holomorphic variation.

Corollary. The bidisk $\mathbf{D} \times \mathbf{D}$ admits no Kähler metric with $\mathrm{HBC}_\omega \leq -1$.