A Kähler–Ricci flow proof of the Wu–Yau Theorem

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A complex manifold (X, g, J, ω) is Kähler if the $(1,1)$–form $\omega := g(J\cdot, \cdot)$ is closed:

$$d\omega = 0.$$

Key examples:

(i) $\mathbb{C}^n$ with the Euclidean metric,

(ii) $\mathbb{P}^n$ with the Fubini–Study metric.

(iii) Complex submanifolds of these.

Key message of the previous talk:

Kähler geometry is particularly successful because of the readily available use of cohomology.

Cohomology makes the Ricci curvature comparitively easier to study since:

The Ricci curvature represents (2π times) the first Chern class of the anti-canonical bundle

$$[\mathrm{Ric}_\omega] \;=\; 2\pi c_1(K_X^{-1}).$$

In particular, if $\mathrm{Ric}_\omega < 0$ on a compact Kähler manifold (X, ω), the canonical bundle K_X is ample.

Recall that a line bundle $\mathcal{L} \to X$ is said to be ample if the sections of a sufficiently high tensor power $\mathcal{L}^{\otimes k}$ furnish a holomorphic embedding

$$\Phi : X \longrightarrow \mathbb{P}^{N_k}.$$

The converse is true by the Aubin–Yau theorem:

Let X be a compact Kähler manifold. Then

$$K_X \text{ ample} \iff \exists \omega \text{ such that } \mathrm{Ric}_\omega < 0.$$

The Ricci flow starting from a Kähler metric ω_0 is given by a family of Riemannian metrics g_t such that

$$\frac{\partial g_t}{\partial t} = -\mathrm{Ric}_{g_t}, \qquad g|_{t=0} = g_0.$$

The Ricci flow preserves the Kähler condition, and the resulting flow is called the Kähler–Ricci flow.

A cohomology class in $H^2_{\mathrm{DR}}(X, \mathbb{R})$ is called a Kähler class if it is represented by a Kähler form.

The set of Kähler classes in the $H^2_{\mathrm{DR}}(X, \mathbb{R})$ form an open convex cone – the Kähler cone.

An important criterion for a cohomology class $[\alpha] \in H^2_{\mathrm{DR}}(X, \mathbb{R})$ being Kähler is given by Demailly–Paun (2004):

The class $[\alpha]$ is Kähler if and only if for all (positive-dimensional) irreducible analytic subvarieties $V \subset X$, the intersection number

$$\int_V \alpha^p > 0.$$

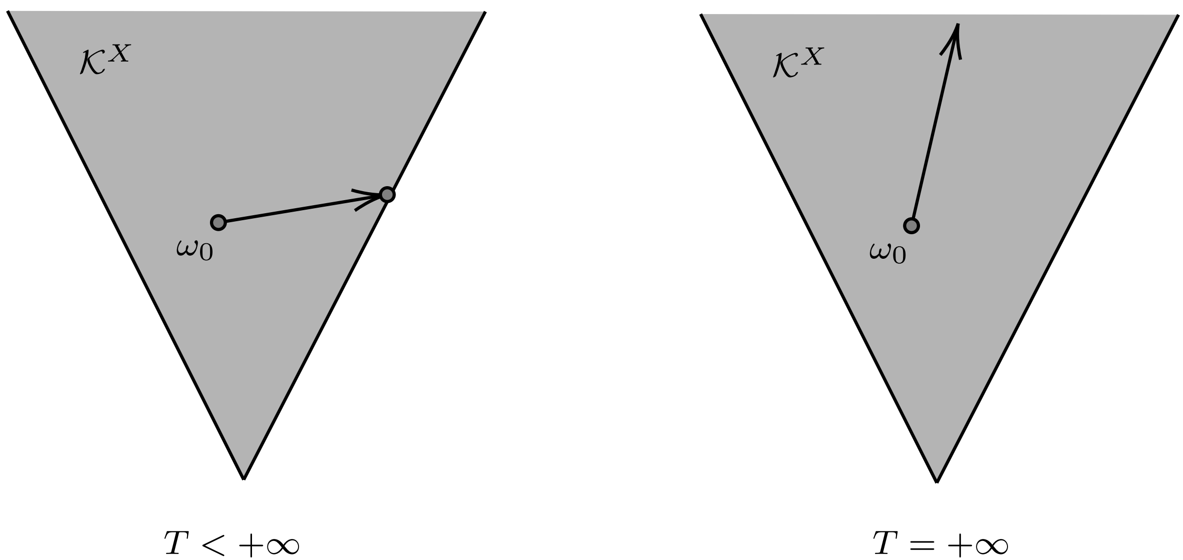
A cohomology class $\alpha \in H^2_{\mathrm{DR}}(X, \mathbb{R})$ on the boundary of the Kähler cone is called a nef class.

Let (X^n, ω_0) be a compact Kähler manifold.

Then the Kähler–Ricci flow has a unique solution ω_t defined on the maximal time interval $[0, T)$, where

$$T \;:=\; \sup\{t > 0 : [\omega_0] + 2\pi t c_1(K_X) \text{ is Kähler}\}.$$

The Kähler–Ricci flow exists for all time $\iff$ the canonical bundle K_X is nef.



Moreover, $[\omega_t] \to 2\pi c_1(K_X)$ as $t \to \infty$.

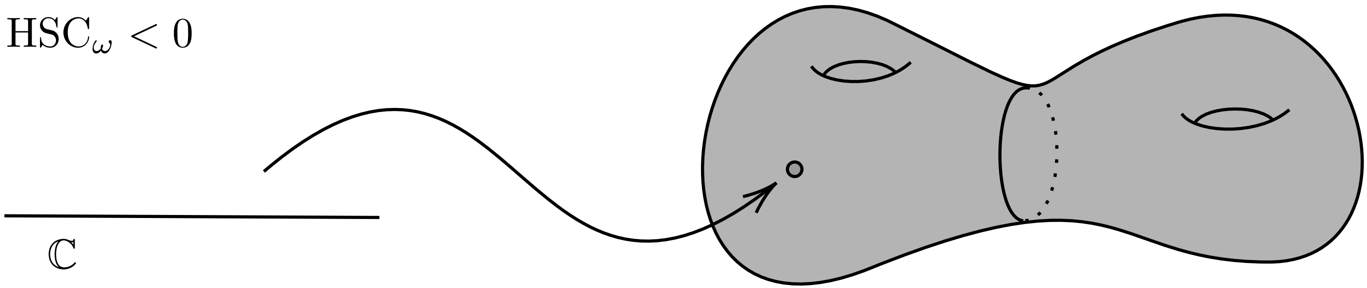
If R denotes the (Riemannian) curvature tensor of a Kähler metric ω, with complex structure J, the holomorphic sectional curvature is given by

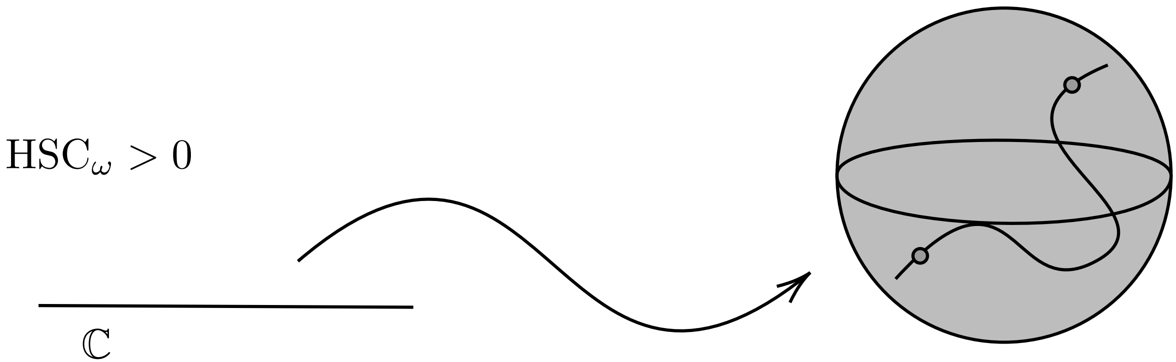
$$\mathrm{HSC}_{\omega}(u) \ := \ \frac{1}{|u|_{\omega}^4} R(u, Ju, u, Ju).$$

In terms of $(1,0)$–vectors $v \in T^{1,0}X$, $v = u - \sqrt{-1}Ju$ the holomorphic sectional curvature reads

$$\mathrm{HSC}_{\omega}(v) \ = \ \frac{1}{|v|_{\omega}^4} \sum_{i,j,k,\ell=1}^{n} R_{i\bar{j}k\bar{\ell}} v_i \overline{v}_j v_k \overline{v}_{\ell}.$$

The holomorphic sectional curvature controls the distortion of holomorphic maps.





A compact Kähler manifold (X, ω) with

($\dagger$) $\text{HSC}_\omega < 0$ is Kobayashi hyperbolic – all holomorphic maps $\mathbb{C} \to X$ are constant.

($\dagger$) $\text{HSC}_\omega > 0$ is rationally connected – any two points lie in the image of a rational curve $\mathbb{P}^1 \to X$.

The Wu–Yau theorem states the following curious relationship between the Ricci curvature and the holomorphic sectional curvature:

If (X, ω) is compact Kähler. Then

$$\mathrm{HSC}_{\omega} < 0 \implies \exists\ \omega_{\varphi} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi \text{ such that } \mathrm{Ric}_{\omega_{\varphi}} < 0.$$

In particular,

$$\mathrm{HSC}_{\omega} < 0 \implies K_X \text{ ample.}$$

– Strategy of the proof –

$\mathrm{HSC}_\omega < 0 \implies$ KRF exists for all time $\iff$ K_X is nef.

The proof is completed by showing that the limiting class is a Kähler class.

Most of the work is establishing the second-order estimate

$$\omega_t \geq C^{-1}\widehat{\omega}.$$

The Demailly–Paun criterion[1] bypasses higher-order estimates.

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[1] A cohomology class $[\alpha]$ is Kähler if and only if for all (positive-dimensional) irreducible analytic subvarieties $V \subset X$, the intersection number

$$\int_V \alpha^p > 0.$$

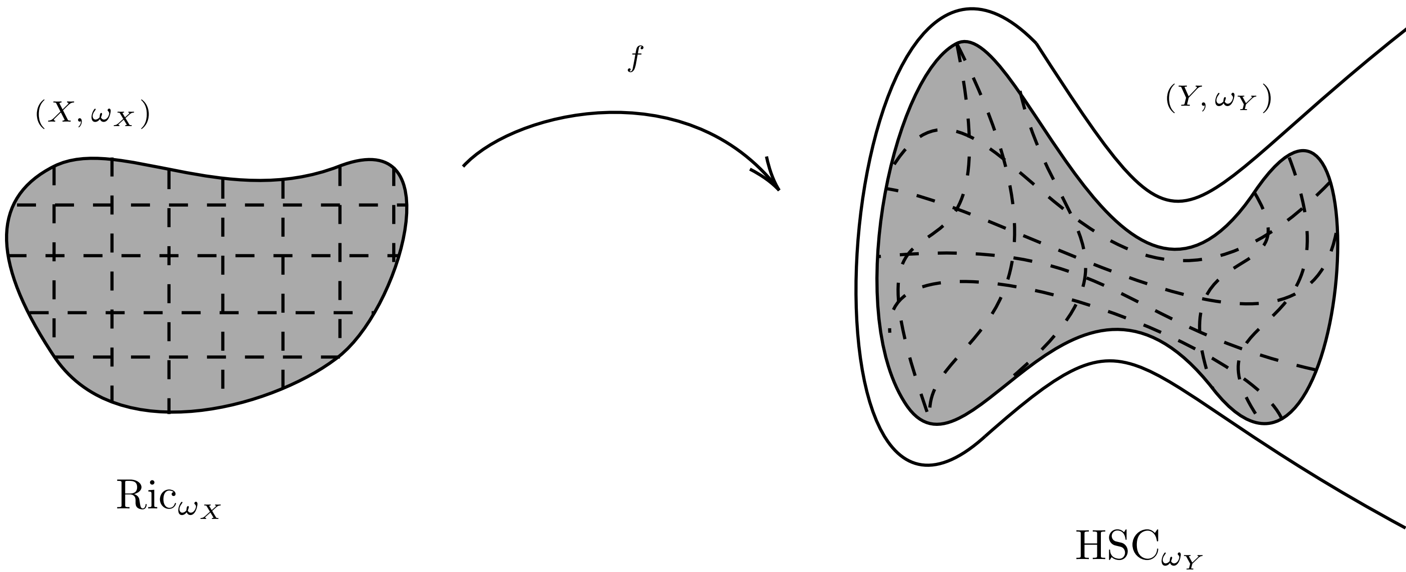
Since $|\partial f|^2 = \mathrm{tr}_{\omega_X}(f^*\omega_Y)$,

$$|\partial f|^2 \leq C \implies \mathrm{tr}_{\omega_X}(f^*\omega_Y) \leq C \implies \omega_X \geq C^{-1} f^*\omega_Y.$$

Hence, we want to estimate $|\partial f|^2$.

– Key technique – Schwarz lemma –

If $f : (X, \omega_X) \longrightarrow (Y, \omega_Y)$ is a holomorphic map. We think of $|\partial f|^2$ as a measure of the elastic tension placed on $f(X)$ as it rests in Y.



The Bochner formula applied to a section $\sigma \in H^0(\mathcal{E})$ of some holomorphic vector bundle $\mathcal{E} \to X$ reads

$$\sqrt{-1}\partial\overline{\partial}|\sigma|^2 = \langle \nabla^{1,0}\sigma, \nabla^{1,0}\sigma \rangle - \sqrt{-1}\langle R^{\mathcal{E}}\sigma, \sigma \rangle.$$

The differential ∂f is a section of the twisted cotangent bundle

$$(T^{1,0}X)^* \otimes f^*T^{1,0}Y.$$

The curvature of the tensor product of bundles splits additively:

$$R^{(T^{1,0}X)^* \otimes f^*T^{1,0}Y} \quad = \quad -R^{T^{1,0}X} \otimes \mathrm{id} + \mathrm{id} \otimes f^*R^{T^{1,0}Y}.$$

The Bochner formula therefore reads:

$$\sqrt{-1}\partial\overline{\partial}|\partial f|^2 = \langle \nabla^{1,0}\partial f, \nabla^{1,0}\partial f\rangle + \sqrt{-1}\langle (R^{T^{1,0}X} \otimes \mathrm{id})\partial f, \partial f\rangle$$
$$-\sqrt{-1}\langle (\mathrm{id} \otimes f^* R^{T^{1,0}Y})\partial f, \partial f\rangle$$

In coordinates:

$$\partial_i \partial_{\overline{j}}(g^{k\overline{\ell}} h_{\gamma\overline{\delta}} f_k^{\gamma} \overline{f_\ell^{\delta}}) = g^{k\overline{\ell}} h_{\gamma\overline{\delta}} f_{ik}^{\gamma} \overline{f_{j\ell}^{\delta}} + R^g_{i\overline{j}k\overline{\ell}} g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f_p^{\alpha} \overline{f_q^{\beta}}$$
$$-g^{p\overline{q}} R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^{\alpha} \overline{f_j^{\beta}} f_p^{\gamma} \overline{f_q^{\delta}},$$

where g, h are the respective metrics underlying ω_X and ω_Y.

Taking the trace of the Bochner formula:

$$g^{i\bar{j}}\partial_i\partial_{\bar{j}}(g^{k\bar{\ell}}h_{\gamma\bar{\delta}}f_k^\gamma\overline{f_\ell^\delta}) \quad = \quad \underbrace{g^{i\bar{j}}g^{k\bar{\ell}}h_{\gamma\bar{\delta}}f_{ik}^\gamma\overline{f_{j\ell}^\delta}}_{=|\nabla\partial f|^2} + \underbrace{g^{i\bar{j}}R^g_{i\bar{j}k\bar{\ell}}g^{k\bar{q}}g^{p\bar{\ell}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta}}_{\text{source curvature term}}$$

$$- \underbrace{g^{i\bar{j}}g^{p\bar{q}}R^h_{\alpha\bar{\beta}\gamma\bar{\delta}}f_i^\alpha\overline{f_j^\beta}f_p^\gamma\overline{f_q^\delta}}_{\text{target curvature term}},$$

The source curvature term

$$g^{i\overline{j}} R^{g}_{i\overline{j}k\overline{\ell}} g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f^{\alpha}_{p} \overline{f^{\beta}_{q}}$$

is controlled by the Ricci curvature.

Since

$$\mathrm{Ric}^{g}_{k\overline{\ell}} \;=\; g^{i\overline{j}} R_{i\overline{j}k\overline{\ell}}.$$

If $\mathrm{Ric}^g \geq -C_1 g + C_2 h$, then

$$\begin{aligned} \mathrm{Ric}^g_{k\overline{\ell}} g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f_p^\alpha \overline{f_q^\beta} &\geq (-C_1 g_{k\overline{\ell}} + C_2 h_{\gamma\overline{\delta}} f_k^\gamma \overline{f_\ell^\delta}) g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f_p^\alpha \overline{f_q^\beta} \\ &\geq -C_1 \mathrm{tr}_{\omega_X}(f^*\omega_Y) + \frac{C_2}{n} \mathrm{tr}_{\omega_X}(f^*\omega_Y)^2, \end{aligned}$$

where the second inequality makes use of Cauchy–Schwarz.

What remains is to understand is the target curvature term

$$-g^{i\bar{j}} g^{p\bar{q}} R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \overline{f_j^\beta} f_p^\gamma \overline{f_q^\delta}.$$

Choose coordinates at a point $p \in X$ and $f(p) \in Y$ such that

$$g_{i\bar{j}}(p) = \delta_{ij}, \qquad \text{and} \qquad h_{\alpha\bar{\beta}}(f(p)) = \delta_{\alpha\beta}.$$

If $f = (f^1, ..., f^n)$, with $f_i^\alpha = \frac{\partial f^\alpha}{\partial z_i}$, the coordinates can be chosen such that

$$f_i^\alpha = \lambda_i \delta_i^\alpha,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = \cdots = 0$, and $r = \text{rank}(\partial f)$.

In these coordinates, we find that

$$g^{i\bar{j}} g^{p\bar{q}} R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \overline{f_j^\beta} f_p^\gamma \overline{f_q^\delta} \quad = \quad \sum_{\alpha,\gamma} R^h_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \lambda_\alpha^2 \lambda_\gamma^2.$$

– A word from our sponsor –

The first attempt to understand this curvature term was given by Yang–Zheng (2015), controlling it by what they call the

Real Bisectional Curvature RBC_ω.

In 2021, I refined this curvature, given an interpretation of the real bisectional curvature as a Rayleigh quotient, and subsequently sharpening the curvature constraint to what I've called the

Second Schwarz Bisectional Curvature $\text{SBC}_\omega^{(2)}$.

A systematic treatment of these curvatures was given in a series paper in 2021–2022, some joint with Kai Tang.

In the Kähler case, the target curvature term is controlled by the holomorphic sectional curvature HSC_ω.

This referred to as Royden's trick:

Let $\xi_1, ..., \xi_\nu$ be othogonal tangent vectors. Let $S(\xi, \overline{\eta}, \zeta, \overline{\omega})$ be a symmetric bi-Hermitian form in the sense that

$$S(\xi, \overline{\eta}, \zeta, \overline{\omega}) = S(\zeta, \overline{\eta}, \xi, \overline{\omega}), \quad \text{and} \quad S(\eta, \overline{\xi}, \omega, \overline{\zeta}) = \overline{S}(\xi, \overline{\eta}, \zeta, \overline{\omega}).$$

If $S(\xi, \overline{\xi}, \xi, \overline{\xi}) \leq -\kappa \|\xi\|^4$, then

$$\sum_{\alpha,\beta} S(\xi_\alpha, \overline{\xi}_\alpha, \xi_\beta, \overline{\xi}_\beta) \leq -\frac{\kappa}{2} \left[\left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2 + \sum_\alpha \|\xi_\alpha\|^4 \right].$$

Further, if $-\kappa \leq 0$, then

$$\sum_{\alpha,\beta} S(\xi_\alpha, \overline{\xi}_\alpha, \xi_\beta, \overline{\xi}_\beta) \leq -\kappa \frac{n+1}{2n} \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2.$$

Hence, we have the Schwarz lemma estimate:

$$\Delta_{\omega_X} \log |\partial f|^2 \quad \geq \quad -C_1 + \frac{C_2}{n} |\partial f|^2 + \frac{\kappa(n+1)}{2n} |\partial f|^2$$

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[2] Recall: $\mathrm{Ric}_{\omega_X} \geq -C_1 \omega_X + C_2 \omega_Y$, $\mathrm{HSC}_{\omega_Y} \leq -\kappa$, and $|\partial f|^2 = \mathrm{tr}_{\omega_X}(f^* \omega_Y)$.

We will apply the Schwarz lemma with: $\omega_X = \omega_t$ (solution to Kähler–Ricci flow), and $\omega_Y = \widehat{\omega}$ (an auxilliary Kähler metric with $\mathrm{HSC}_{\widehat{\omega}} \leq -\kappa$).

Moreover, we will want a parabolic Schwarz lemma in the sense that we want an estimate on

$$(\partial_t - \Delta_{\omega_t}) \log \mathrm{tr}_{\omega_t}(\widehat{\omega}).$$

From the Kähler–Ricci flow, we obtain the following parabolic Schwarz lemma:

$$(\partial_t - \Delta_{\omega_t}) \log \operatorname{tr}_{\omega_t}(\widehat{\omega}) \quad \leq \quad 1 - \frac{\kappa(n+1)}{2n} \operatorname{tr}_{\omega_t}(\widehat{\omega}).$$

Theorem. Let (X, ω) be a compact Kähler manifold with $\mathrm{HSC}_{\omega} < 0$. Then K_X is nef.

It suffices to show that the Kähler–Ricci flow ω_t, starting at some initial Kähler metric ω_0, exists for all time.

By the parabolic Schwarz lemma:

$$(\partial_t - \Delta_{\omega_t})(\log \mathrm{tr}_{\omega_t}(\widehat{\omega}) - t) \quad \leq \quad -\frac{\kappa(n+1)}{2n}\mathrm{tr}_{\omega_t}(\widehat{\omega}).$$

For any $t \in [0, T_0)$, by the maximum principle:

$$\operatorname{tr}_{\omega_t}(\widehat{\omega}) \;\leq\; e^t \max_X \operatorname{tr}_{\omega_0}(\widehat{\omega}) \;\leq\; e^{T_0} \max_X \operatorname{tr}_{\omega_0}(\widehat{\omega}) \;=:\; C.$$

Hence, we have a uniform constant $C > 0$ such that

$$\omega_t \;\geq\; C^{-1}\widehat{\omega}.$$

Let $V \subset X$ be any positive-dimensional irreducible subvariety with $p := \dim(V)$. Then

$$\int_V \omega_{T_0}^p \quad = \quad \lim_{t \to T_0} \int_V \omega_t^p \;\geq\; C^{-p} \int_V \widehat{\omega}^p \;>\; 0.$$

By the Demailly–Paun characterization of the Kähler cone, this implies that $[\omega_{T_0}]$ is a Kähler class.

Let (X, ω) be a compact Kähler manifold with $\text{HSC}_\omega \leq -\kappa < 0$. Then K_X is ample.

If (X, ω) is compact Kähler with $\text{HSC}_\omega < 0$, the Kähler–Ricci flow exists for all time from the previous theorem.

It suffices to show that the limit class is a Kähler class.

But we can just apply the same argument:

$$\int_V \omega_\infty^p \;=\; \lim_{t\to\infty} \int_V \omega_t^p \;\geq\; C^{-p} \int_V \widehat{\omega}^p \;>\; 0.$$

Since $2\pi c_1(K_X) = [\omega_\infty]$, it follows that $c_1(K_X) > 0$ and therefore, K_X is ample.

– Most general forms of the Wu–Yau theorem –

There are two bottlenecks to generating theorems of Wu–Yau-type:

(1.) The Schwarz lemma.

(2.) The existence of an ambient Kähler structure.

The furthest progress on the Schwarz lemma bottleneck was made by my (2021) Schwarz lemma paper:

Theorem. (B.-). Let X be a compact Kähler manifold which supports a Hermitian metric with negative second Schwarz bisectional curvature $\text{SBC}_{\widehat{\omega}}^{(2)} < 0$. Then K_X is ample.

Question. Can we $\text{SBC}_{\widehat{\omega}}^{(2)} < 0$ be relaxed to $\text{HSC}_{\widehat{\omega}} < 0$?

The furthest progress on the ambient Kähler condition is made by Man-Chun Lee (2020):

Theorem. Let (X, ω_0) be a compact Hermitian manifold with $\text{HBC}_{\omega_0} \leq 0$. Let ω_t be a solution of

$$\frac{\partial}{\partial t}\omega_t = -\text{Ric}^{(2)}_{\omega_t}, \qquad \omega_t|_{t=0} = \omega_0.$$

If $\text{Ric}^{(1)}_{\omega} \leq 0$ everywhere and $\text{Ric}^{(1)}_{\omega_t} < 0$ at some point, then K_X is ample.

Conjecture. Let (X, ω) be a compact Hermitian manifold with $\text{HSC}_{\omega} < 0$. Then K_X is ample.

Question. Let (X, ω) be a compact Hermitian manifold with $\text{HSC}_{\omega} < 0$. Is $\text{Ric}_{\omega}^{(1)} < 0$?