MATH2021 SUMMARY NOTES

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Videos to accompany this material may be found here: https://www.youtube.com/channel/UCkDicHMdLaJQ3OIly7B6wug

1. Tutorial 1: Arc-Length Parametrization

Why parametrize a curve in terms of its arc length?

To answer this, let $\alpha(t)$ be a parametrized curve. Intuitively, $\alpha(t)$ describes the motion of a particle as a function of time t. If we reparametrize the time, then the only extra information we get about the path is the velocity of the particle, more or less. The shape of the curve, i.e., the graph of $\alpha(t)$ does not change.

Typically, as differential geometers, we want to study properties of the graph of $\alpha(t)$ that are intrinsic, and do not depend on the choice of parametrization: for example, we want to understand the curvature of the graph, i.e., is the curve flat like $\mathbb{R}^2$, positively curved like the sphere $\mathbb{S}^2$, or negatively curved like a saddle?

To compute these intrinsic properties, it is often most convenient to parametrize the curve (so that we can do computations). The problem with this, however, is that it attaches additional velocity-type data that we mentioned above, and we cannot guarentee that whatever we compute is in fact intrinsic to the curve.

One special reparametrization, however, does ensure this: the arc-length parametrization. A useful fact to know is that arc length parametrization corresponds to a parametrization which has unit velocity everywhere: Let

$$s = \int_0^t |\alpha'(x)| dx$$

be the arc length parametrization. Then by the fundmantal theorem of calculus,

$$\frac{ds}{dt} = |\alpha'(t)|,$$

and in particular,

$$\frac{dt}{ds} = \frac{1}{|\alpha'(t)|}.$$

So by the chain rule:

$$\alpha'(s) \quad = \quad \frac{d\alpha}{ds} \quad = \quad \frac{d\alpha}{dt}\frac{dt}{ds} \quad = \quad \alpha'(t)\frac{1}{|\alpha'(t)|},$$

which has length 1.

2. **Tutorial 2: Curve Integrals**

Coming soon.

3. **Tutorial 3: Conservation**

Definition 3.1. A vector field $\mathbf{F}(x, y)$ is said to be **conservative** if $\mathbf{F} = \nabla f$ for some function f.

Theorem 3.2. If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Of course, we need to assume that P and Q have continuous first-order partial derivatives for this to make sense.

The following theorem essentially tells us that the converse is also true:

Theorem 3.3. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D.[1] Suppose P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

on D, then $\mathbf{F}$ is a conservative vector field.

Theorem 3.4. Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D.

[1] For example, let D be convex.

Example 3.5. Determine whether or not these vector fields are conservative:

(a) $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$.

Proof. Here $P(x, y) = x - y$ and $Q(x, y) = x - 2$. By Theorem 3.3, we need to check whether $P_y = Q_x$. To this end, we note that

$$P_y = -1, \qquad Q_x = 1.$$

So $\mathbf{F}$ is not a conservative vector field. □

(b) $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$.

Proof. Here $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. Again, we need to check whether $P_y = Q_x$:

$$P_y = 2x, \qquad Q_x = 2x,$$

so $\mathbf{F}$ is a conservative vector field. □

Theorem 3.6. (Fundamental theorem of line integrals). Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} \;=\; f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

4. Tutorial 4: Double Integrals

Example 4.1. Evaluate the integral $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution. Always draw the region of integration D first.[2] The point of intersection between these curves occurs at $x = 0$ and $x = 1$. So the x bounds are $0 \leq x \leq 1$. The y–integral needs to have variable bounds, and these are given by $2x^2 \leq y \leq 1 + x^2$. So we simply need to evaluate the integral

$$\begin{aligned} \int_0^1 \int_{2x^2}^{1+x^2} (x + 2y) dy dx \;&=\; \int_0^1 \left[xy + y^2\right]_{2x^2}^{1+x^2} dx \\ &=\; \frac{32}{15}. \end{aligned}$$

Details of the computation, as well as a computational hint, are provided in the video. □

Polar coordinates. Here, we make the change of variables $x = r\cos\vartheta$ and $y = r\sin\vartheta$. Observe that as a consequence of the pythagorean identity, we have

$$x^2 + y^2 = r^2.$$

[2]This is done in the video which accompanies this example: https://www.youtube.com/watch?v=KUxCzXpuKjO&

Theorem 4.2. If f is continuous on a polar region given by $a \leq r \leq b$ and $\vartheta_1 \leq \vartheta \leq \vartheta_2$, then

$$\iint_D f(x,y)dA = \int_{\vartheta_1}^{\vartheta_2} \int_a^b f(r\cos\vartheta, r\sin\vartheta) r dr d\vartheta.$$

Remark 4.3. Let us address why $dA = rdrd\vartheta$. Here, we have $x = r\cos\vartheta$ and $y = r\sin\vartheta$. Therefore,

$$\frac{\partial x}{\partial r} = \cos\vartheta, \quad \frac{\partial x}{\partial \vartheta} = -r\sin\vartheta, \quad \frac{\partial y}{\partial r} = \sin\vartheta, \quad \frac{\partial y}{\partial \vartheta} = r\cos\vartheta.$$

To determine the extent to which area is distorted when we change to polar coordinates, we need to look at the determinant of the Jacobian matrix:

$$\begin{aligned} \det\begin{pmatrix} \partial_r x & \partial_\vartheta x \\ \partial_r y & \partial_\vartheta y \end{pmatrix} &= \det\begin{pmatrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \end{pmatrix} \\ &= r\cos^2\vartheta + r\sin^2\vartheta \\ &= r. \end{aligned}$$

Since

$$dA = \det\begin{pmatrix} \partial_r x & \partial_\vartheta x \\ \partial_r y & \partial_\vartheta y \end{pmatrix} drd\vartheta,$$

we see that $dA = rdrd\vartheta$.

Example 4.4. Evaluate $\iint_D (3x + 4y^2)dA$, where D is the region bounded by the first two quadrants of $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution. Always draw the region of integration first. Here, the radius is bounded between $1 \leq r \leq 4$, and the angle ranges from $0 \leq \vartheta \leq \pi$. Hence, using polar coordinates, we see that

$$\begin{aligned} \iint_D (3x + 4y^2)dA &= \int_0^\pi \int_1^2 (3r\cos\vartheta + 4r^2\sin^2\vartheta) r dr d\vartheta \\ &= \int_0^\pi \int_1^2 (3r^2\cos\vartheta + 4r^3\sin^2\vartheta) dr d\vartheta \\ &= \frac{15\pi}{2}. \end{aligned}$$

□

5. Week 7 – Triple Integrals, Gauss' and Stokes' Theorems

Triple integrals are very similar to double integrals. Let us look at a basic example of a triple integral computation:

Example 7.1. Evaluate $\iiint_E \sqrt{x^2+z^2}dV$, where E is the region bounded by $y = x^2 + z^2$ and the plane $y = 4$.

Solution. The region E is given by a paraboloid whose central axis is along the y–axis, which is capped off at $y = 4$. We observe that $-2 \leq x \leq 2$, $x^2 \leq y \leq 4$ and

$$-\sqrt{y-x^2} \leq z \leq \sqrt{y-x^2}.$$

Hence, the integral is given by

$$\iiint_E \sqrt{x^2+z^2}dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}dzdydx.$$

Given the circular nature of the region in the x and z coordinates, we set $x = r\cos\vartheta$ and $z = r\sin\vartheta$. Then $x^2 + z^2 = r^2$ and □