Chapter 1

Trigonometry.

In this chapter we look at the techniques used to solve elementary trigonometric equations. This begins with a discussion of the functions $f(x) := \cos x$, $f(x) := \sin x$ and $f(x) = \tan x$. We then move on to the associated reciprocal and inverse functions for these types of curves.

1.1 Conversion between Radians and Degrees

The first thing we look at is converting degrees to radians. Radians are the primary unit of angles that mathematicians use, not degrees. The formula for converting degrees to radians is given by

$$x \longmapsto \frac{\pi}{180}x.$$

The corresponding formula that converts radians to degrees is given by

$$x \longmapsto \frac{180}{\pi}x.$$

Example 3.1.1. Convert $275°$ to radians.

Proof. Using the above formula, we simply observe that

$$275° = \frac{\pi}{180}(275) = \frac{55}{36}\pi.$$

□

Example 3.1.2. Converse $\frac{\pi}{3}$ into degrees.

Proof. Using the above formula, we simply observe that

$$\frac{\pi}{3} = \frac{\pi}{3} \cdot \frac{180}{\pi} = \frac{180^\circ}{3} = 60^\circ.$$

$\square$

Exercises

Q1. Convert the following to radians.

a. 360°.

b. 200°.

c. 160°.

d. 670°.

e. 120°.

f. 165°.

g. 180°.

h. 720°.

Q2. Convert the following to degrees.

a. π.

b. $\frac{\pi}{2}$.

c. $\frac{3\pi}{2}$.

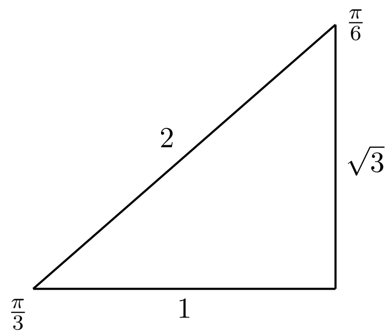
d. $\frac{4\pi}{7}$.

e. $\frac{3\pi}{8}$.

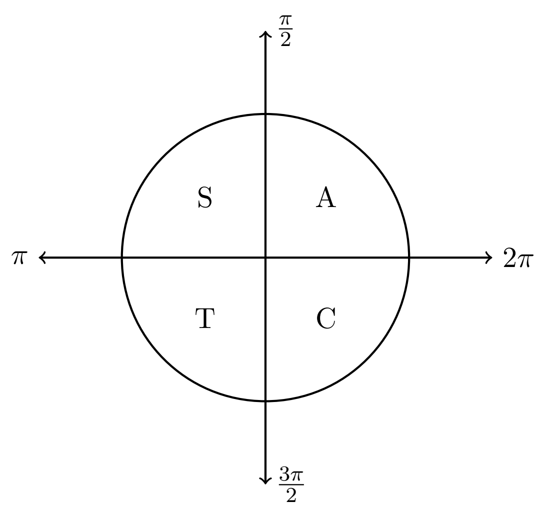
f. $\frac{4\pi}{3}$.

1.2 Exact Values

Our weapon of choice for determining the exact values of trigonometric functions with arguments $\vartheta = \frac{\pi}{3}$ and $\frac{\pi}{6}$ is the following triangle



and the following diagram depicting the quadrants of the unit circle



Example 3.2.1. Determine the exact value of $\sin\left(\frac{4\pi}{3}\right)$.

Proof. The magnitude of $\sin\left(\frac{4\pi}{3}\right)$ is the same as the magnitude of $\sin\left(\frac{\pi}{3}\right)$, so let us first determine what $\sin\left(\frac{\pi}{3}\right)$ is. Using the above triangle, we see that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. The multiplication by 4 which rotates $\frac{\pi}{3}$ to $\frac{4\pi}{3}$ just tells us which quadrant we are in. Notice that if we ignore the π for the moment, we are considering the number $\frac{4}{3}$. It is easy to see that

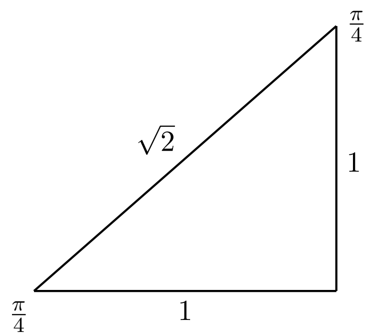
$$1 < \frac{4}{3} < \frac{3}{2} \implies \pi < \frac{4\pi}{3} < \frac{3\pi}{2}.$$

So we are in the third quadrant. Since the sin function is negative in the third quadrant, it follows that

$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$$

□

To determine the exact values of trigonometric functions with arguments $\vartheta = \frac{\pi}{4}$, we could use the triangle



Instead however, it is often quite easy just to remember that

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ \text{and} \ \tan\left(\frac{\pi}{4}\right) = 1.$$

Example 3.2.2 Determine the exact value of $\tan\left(\frac{3\pi}{4}\right)$.

Proof. Recall that the magnitude of $\tan\left(\frac{3\pi}{4}\right)$ is equal to the magnitude of $\tan\left(\frac{\pi}{4}\right) = 1$. Moreover, ignoring the π for the moment, we see that

$$\frac{1}{2} < \frac{3}{4} < 1 \implies \frac{\pi}{2} < \frac{3\pi}{4} < \pi.$$

So we are in the second quadrant. Since the tan function is negative in the second quadrant, we see that

$$\tan\left(\frac{3\pi}{4}\right) = -1.$$

□

To determine the exact values of trigonometric functions with arguments $\vartheta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π, we make the natural identification of $x = \cos\vartheta$ and $y = \sin\vartheta$. If we then look at $\vartheta = \frac{\pi}{2}$, we notice that this has coordinate $(0, 1)$ on the unit circle. Since $x = \cos\vartheta$ and $y = \sin\vartheta$, it follows that

$$\cos\left(\frac{\pi}{2}\right) = 0, \text{ and } \sin\left(\frac{\pi}{2}\right) = 1.$$

Exercises

Q1. Using the exact value triangles, determine the following.

a. $\sin\frac{\pi}{4}$.

b. $\cos\frac{3\pi}{4}$.

c. $\sin\frac{\pi}{3}$.

d. $\cos\frac{5\pi}{6}$.

e. $\sin\frac{7\pi}{6}$.

f. $\cos\frac{\pi}{4}$.

Q2. By looking at the quadrants, determine the following.

a. $\sin\frac{\pi}{2}$.

b. $\cos 0$.

c. $\tan\frac{3\pi}{2}$.

d. $\cos\pi$.

e. $\sin\pi$.

f. $\tan 0$.

g. $\cos 2\pi$.

h. $\cos\frac{\pi}{2}$.

i. $\sin\frac{3\pi}{2}$.

j. $\tan 4\pi$.

Q3. For $0 < \vartheta < \frac{\pi}{2}$, determine the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$.

b. $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

c. $\tan^{-1}(\sqrt{3})$.

d. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Q4. For $0 < \vartheta < \pi$, determine the following.

a. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

b. $\tan^{-1}(-1)$.

c. $\sin^{-1}\left(\frac{1}{2}\right)$.

d. $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

Q5. Suppose $\cos\vartheta = \frac{1}{5}$. Determine the exact value of

a. $\sin\vartheta$. b. $\tan\vartheta$.

Q6. Suppose $\tan\vartheta = \sqrt{3} + 1$. Determine the exact value of

a. $\cos\vartheta$. b. $\sin\vartheta$.

1.3 Solving Elementary Trigonometric Equations

In this section we use our knowledge of exact values to solve elementary trigonometric equations.

Example 3.3.1. Solve the equation $\sin x = \frac{\sqrt{3}}{2}$ for $0 \leq x \leq 2\pi$.

Proof. We first determine the reference angle, it is clear that

$$x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

Looking at the quadrants however, we also know that sin is positive in the first and second quadrant. To determine the angle that lies in the second quadrant, we go from the horizontal. Hence, we see that

$$x = 0 + \text{RA}, \pi - \text{RA} = 0 + \frac{\pi}{3}, \pi - \frac{\pi}{3} = \frac{\pi}{3}, \frac{2\pi}{3},$$

where RA of course refers to the *reference angle.* □

Example 3.3.2. Solve the equation $\cos 2x = -\frac{1}{\sqrt{2}}$ for $0 \leq x \leq 2\pi$.

Proof. We first determine the reference angle, it is clear that

$$2x = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

Looking at the quadrants however, we know that cos is negative in the second and third quadrant. Again, going from the horizontal, we see that

$$\begin{aligned} 2x &= \pi - \text{RA}, \pi + \text{RA}, 3\pi - \text{RA}, 3\pi + \text{RA} \\ &= \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, 3\pi + \frac{\pi}{4} \\ &= \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \\ \therefore\ x &= \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}. \end{aligned}$$

□

Notice that we had to go all the way to 4π since

$$0 \leq x \leq 2\pi \implies 0 \leq 2x \leq 4\pi.$$

If we wanted to solve $\cos 5x = -\frac{1}{\sqrt{2}}$ for $0 \leq x \leq 2\pi$, we would need to go all the way to 10π.

Exercises

Q1. For $x \in [0, 2\pi]$, solve the following equations.

a. $\sin x = \frac{1}{\sqrt{2}}$.

b. $\cos x = -\frac{1}{2}$.

c. $\sin x = \frac{1}{2}$.

d. $\cos x = \frac{\sqrt{3}}{2}$.

e. $\sin x = -\frac{\sqrt{3}}{2}$.

f. $\tan x = 1$.

g. $\tan x = -\sqrt{3}$.

h. $\tan x = \frac{1}{\sqrt{3}}$.

Q2. For $x \in [0, 2\pi]$, solve the following equations.

a. $\sin 2x = \frac{\sqrt{3}}{2}$.

b. $\cos 2x = \frac{\sqrt{2}}{2}$.

c. $\tan 3x = 1$.

d. $\sin \frac{x}{2} = -\frac{1}{2}$.

e. $\tan 2x = -\sqrt{3}$.

f. $\cos 4x = -1$.

g. $\tan x = 0$.

h. $\cos \frac{x}{3} = 0$.

i. $\sin x = 1$.

j. $\tan 2x = -1$.

Q3. Solve the following equation for $x \in [0, 2\pi]$.

$$\sin x = \sqrt{3} \cos x.$$

Q4. Solve the following equation for $x \in [0, 2\pi]$.

$$\sin 2x = \cos 2x.$$

Q5. Solve the following equation for $x \in [0, 2\pi]$.

$$\sin 2x = \sqrt{3} \cos 2x.$$

Q6. Solve the following equation for $0 \leq x \leq 2\pi$.

$$\sin^2 x + 2 \sin x = \frac{5}{4}.$$

Q7. Find the general solution to the equation

$$\sin (3x - \pi) = \frac{1}{\sqrt{2}}.$$

Q8. Find the general solution to the equation

$$\cos\left(\frac{x}{2}\right) = 1.$$

Q9. Ishan only likes to go surfing when the tides are at a height of 3 metres. Ishan knows that the tide follows the following trigonometric relation

$$\mathscr{F}(t) = \frac{3}{2}\cos\left(\frac{\pi}{3}t\right) + \frac{3}{2}.$$

If t is measured in hours, what times of the day can Ishan go surfing?

Q10. On a local beach, the time between high tide and low tide is 6 hours. The average depth of water at a point in the beach is 4 metres and at high tide the depth is 5 metres.

a. If a boat requires a depth of 4 metres of water in order to sail, how many hours before noon can it enter the point in the beach and by what time must it leave before it is to be left stranded?

b. If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point in the beach and by what time must it leave before it is to be left stranded?

1.4 Sketching of Trigonometric Functions.

The general form of the trigonometric equations are

$$\begin{aligned} f(x) &= a\sin(b(x-c))+d, \\ f(x) &= a\cos(b(x-c))+d, \\ f(x) &= a\tan(b(x-c))+d, \end{aligned}$$

where $b, c, d \in \mathbb{R}$ and $a \in \mathbb{R}\backslash\{0\}$.

† The number a corresponds to a dilation from the x-axis by factor a.

† The number b corresponds to a dilation from the y-axis by factor $1/b$.

† If $a < 0$, this corresponds to a reflection across the x-axis.

† If $b < 0$, this corresponds to a reflection across the y-axis.

† The number c corresponds to a translation along the x-axis to the right.

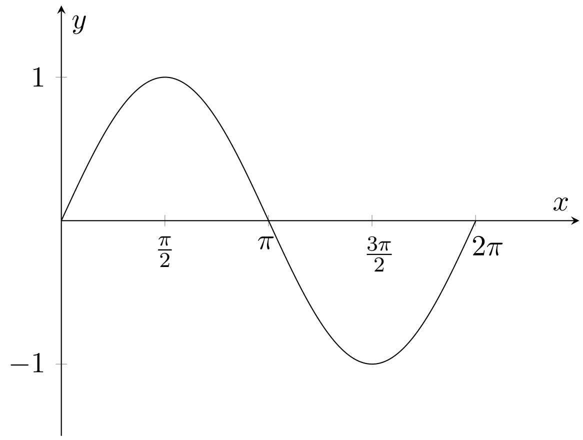
† The number d corresponds to a translation along the y-axis in the upward direction.

You will recall from Chapter 1 that transformations are always given in the order of dilations, reflections and translations (DRT).

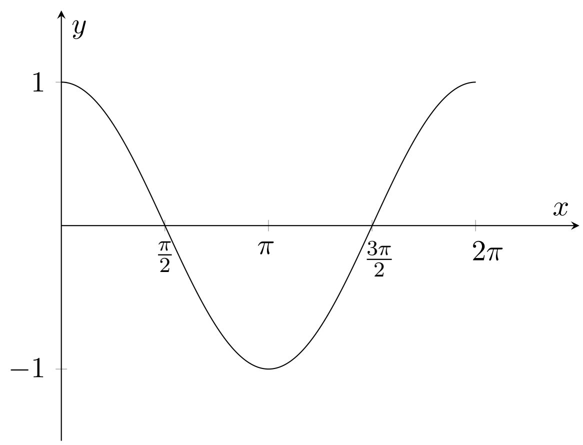
Example 3.4.1. Describe the transformations required to map $f(x) = \sin x$ to $\widetilde{f}(x) = \frac{1}{2}\sin(\pi - x) - \frac{3}{4}$.

Proof. Dilating by factor $\frac{1}{2}$ from the x-axis maps $f(x) = \sin x$ to $f_1(x) = \frac{1}{2}\sin x$. Reflecting across the y-axis maps $f_1(x) = \frac{1}{2}\sin x$ to $f_2(x) = \frac{1}{2}\sin(-x)$. Translating by π units to the left maps $f_2(x) = \frac{1}{2}\sin(-x)$ to $f_3(x) = \frac{1}{2}\sin(\pi - x)$. Finally, translating by $\frac{3}{4}$ units in the negative y-direction maps $f_3(x) = \frac{1}{2}\sin(\pi - x)$ to $\widetilde{f}(x) = \frac{1}{2}\sin(\pi - x) - \frac{3}{4}$. □

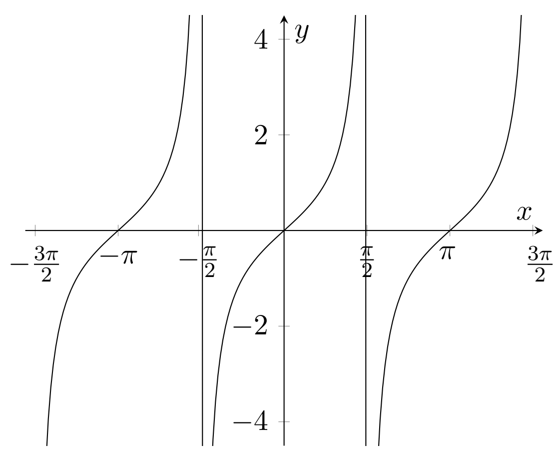
The general shape of the curve $f(x) = \sin x$ is given by



The general shape of the curve $f(x) = \cos x$ is given by

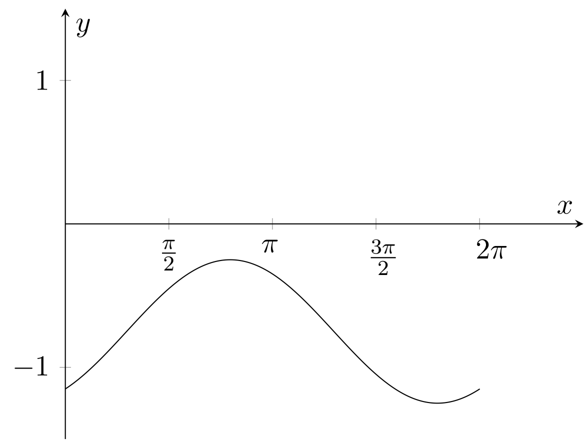


The general shape of the curve of $f(x) = \tan x$ is given by



Example 3.4.2. Sketch the function $f(x) = \frac{1}{2}\sin(\pi - x) - \frac{3}{4}$ on the domain $0 \le x \le 2\pi$.

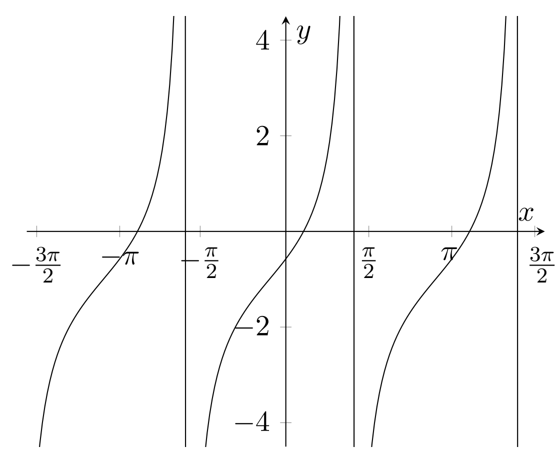
Proof. Let us observe that the function f is obtained from $g(x) := \sin(x)$ by dilating by factor $\frac{1}{2}$ from the x-axis, reflecting about the y-axis, translating by π units to the right and then $\frac{3}{4}$ units down. We therefore see that



□

Example 3.4.3. Sketch the function $f(x) = \frac{4}{3}\tan\left(x + \frac{\pi}{3}\right) - 1$ on the domain $-\frac{3\pi}{2} < x < \frac{3\pi}{2}$.

Proof. Let us observe that the function f is obtained from $g(x) := \tan x$ by dilating by $\frac{4}{3}$ from the x-axis, translating by $\frac{\pi}{3}$ units to the left and then translating by 1 unit down. Hence we see that



□

Exercises

Q1. Let $f(x) = \cos x$. Describe the transformations required to map $f(x)$ to

a. $\widetilde{f}(x) = 2\cos(3x - \pi) + 1$.

b. $\widetilde{f}(x) = \frac{1}{5}\cos\left(\frac{1}{2}\pi - x\right) - 4$.

Q2. Let $f(x) = \sin x$. Describe the transformations required to map $f(x)$ to

a. $\widetilde{f}(x) = \frac{4}{5}\sin(\pi - x) + \frac{1}{2}$.

b. $\widetilde{f}(x) = \sqrt{2}\sin(-x)$.

Q3. Let $f(x) = \tan x$. Describe the transformations required to map $f(x)$ to

a. $\widetilde{f}(x) = \frac{3}{5}\tan\left(x - \frac{\pi}{2}\right) - 3$.

b. $\widetilde{f}(x) = -\tan(-x)$.

Q4. Describe the transformations required to map $f(x) = 3 - \sqrt{3}\sin(-x)$ to $\widetilde{f}(x) = \sin(x)$.

Q5. Describe the transformations required to map

$$f(x) = \frac{3}{7} - \frac{\sqrt{5}}{3}\tan(-x + \pi^2)$$

to $\widetilde{f}(x) = \tan x$.

Q6. Describe the transformations required to map $f(x) = \sin x$ to $\widetilde{f}(x) = \cos x$.

Q7. Describe the transformations required to map $f(x) = \frac{3}{5}\cos\left(4x - \frac{\pi}{3}\right)$ to

$$\widetilde{f}(x) = -\frac{1}{6} + \frac{4}{7}\cos\left(\pi - 7x\right).$$

Q8. Graph the following functions for $0 \leq x \leq 2\pi$, stating all relevant features.

a. $f(x) = 3\sin(x)$

b. $f(x) = \frac{4}{3}\cos(2x)$.

c. $f(x) = \frac{3}{7}\sin(6x)$.

d. $f(x) = \frac{3}{5}\cos\left(\frac{x}{2}\right)$.

Q9. Graph the following functions for $0 \leq x \leq 2\pi$, stating all relevant features.

a. $f(x) = 2 + \cos(x)$.

b. $f(x) = \frac{1}{3} - \sin(2x)$.

c. $f(x) = \frac{3}{5}\cos(4x) - \frac{3}{5}$.

d. $f(x) = \frac{1}{2}\sin(-x) - 1$.

Q10. Graph the following function for $-\pi \leq x \leq \pi$, stating all relevant features.

a. $f(x) = 2\tan(-x)$.

b. $f(x) = \frac{3}{5}\tan(2x)$.

c. $f(x) = 1 + \tan(x)$.

d. $f(x) = 2 - 4\tan(4 - x)$.

e. $f(x) = \frac{1}{3} + \sqrt{3}\tan(-4x)$.

f. $f(x) = \frac{\sqrt{3}}{1+\sqrt{2}}\tan(1 - \sqrt{2}x)$.

Q11. Graph the following function for $-2\pi \leq x \leq 2\pi$, stating all relevant features.

a. $f(x) = \sin|x|$.

b. $f(x) = -3|\cos x| + 1$.

c. $f(x) = |\tan x| - 4$.

d. $f(x) = \sqrt{2}\,|\sin|x|| - 1$.

Q12. Jim, the manager of a reservoir and its catchment area has notes that the inflow of water into the reservoir is very predictable and in fact models the inflow with a curve of the form

$$\mathcal{W}(t) = \alpha\sin(\eta t) + \beta,$$

where $\alpha, \beta, \eta, \varepsilon \in \mathbb{R}$. Jim has the following data,

† The average inflow is 100,000 m^3/day.

† The minimum daily flow is 80, 000 m^3/day.

† The maximum daily flow is 120, 000 m^3/day, and this occurs on the first of May, $t = 121$.

With this knowledge,

a. Find the values of α, β, and η.

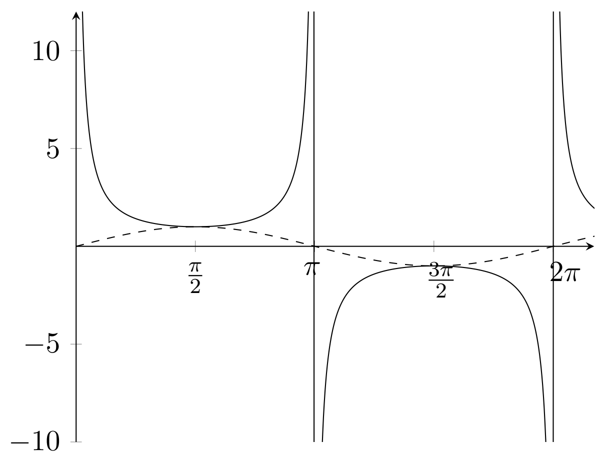
b. Find the times of year when the inflow per day is 90,000 m^3/day.

1.5 Reciprocal Trigonometric Functions

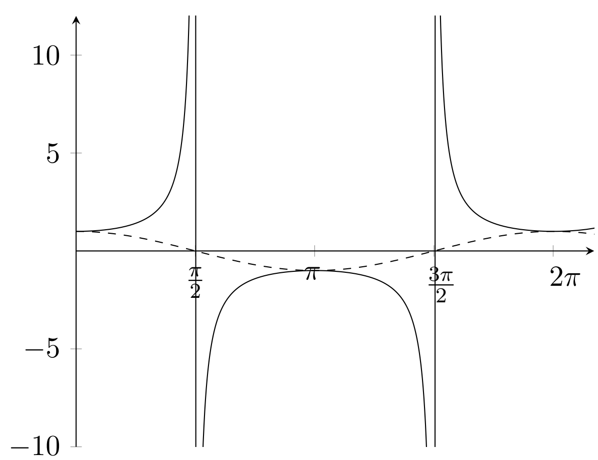
In this section, we study the functions

$$\csc x := \frac{1}{\sin x}, \quad \sec x := \frac{1}{\cos x}, \text{ and } \cot x := \frac{\cos x}{\sin x}.$$

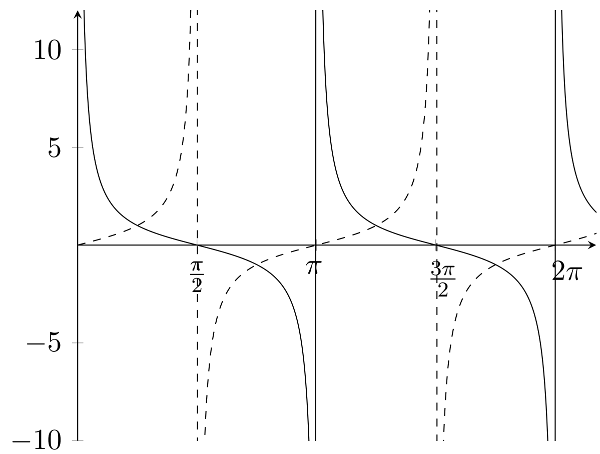
The general shape of the curve $f(x) = \csc x$ is given by



The general shape of the curve $f(x) = \sec x$ is given by



The general shape of the curve $f(x) = \cot x$ is given by



Example 3.5.1. Find the following exact values

a. $\sec\left(\frac{\pi}{3}\right)$.

Proof. We simply observe that

$$\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos(\frac{\pi}{3})} = \frac{1}{\frac{1}{2}} = 2.$$

□

b. $\cot\left(\frac{\pi}{6}\right)$.

Proof. We simply observe that

$$\cot\left(\frac{\pi}{6}\right) = \frac{1}{\tan(\frac{\pi}{6})} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}.$$

□

c. $\csc\left(\frac{\pi}{2}\right)$.

Proof. We simply observe that

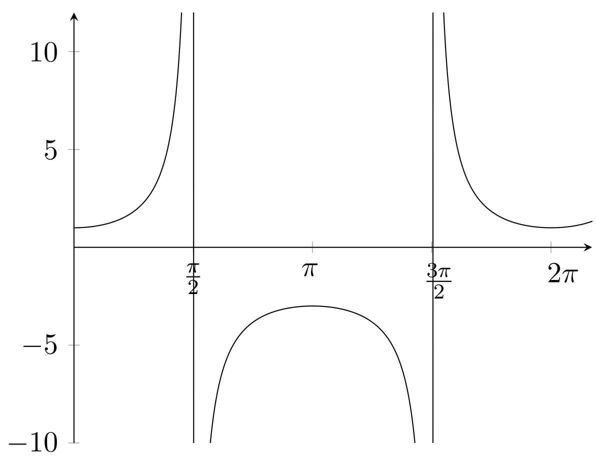
$$\csc\left(\frac{\pi}{2}\right) = \frac{1}{\sin(\frac{\pi}{2})} = \frac{1}{1} = 1.$$

□

Example 3.5.2. Sketch the following curves, stating all relevant features.

a. $f(x) = 2\sec(x) - 1$.

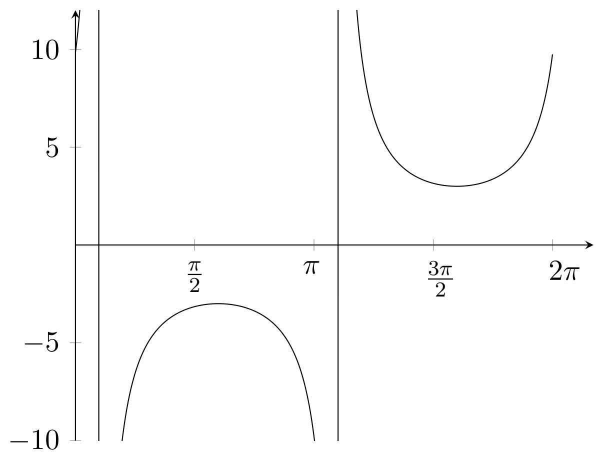
Proof. The function f is obtained from $g(x) := \sec(x)$ by dilating by factor 2 from the x-axis and translating by 1 unit down. It therefore follows that



□

b. $f(x) = 3\csc\left(x - \frac{\pi}{3}\right)$.

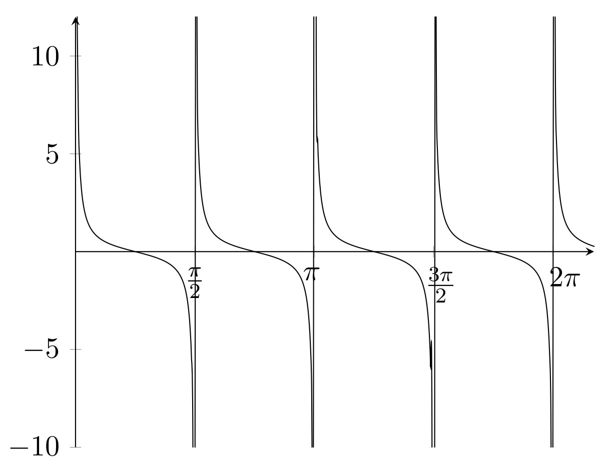
Proof. The function f is obtained from $g(x) := \csc(x)$ by dilating by factor 3 from the x-axis and translating by $\frac{\pi}{3}$ units to the right. The resulting curve is given by



∎

c. $f(x) = \frac{1}{2}\cot(2x)$.

Proof. The function f is obtained from $g(x) := \cot(x)$ by dilating by $\frac{1}{2}$ from the x-axis and by dilating by $\frac{1}{2}$ from the y-axis. We therefore see that



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⚠**Remark 3.5.3**. It is often claimed that $\cot(x) := \frac{1}{\tan(x)}$. We make it a point to argue that this is not correct. Indeed, if we consider the function $f(x) := \tan(x)$, then the domain of f is

$$\mathcal{D}(f) = \left\{ x \in \mathbb{R} \;\middle|\; x \neq \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z} \right\}.$$

That is, the function f is not permitted to input odd multiples of π, since $\tan(x) := \frac{\sin(x)}{\cos(x)}$ and $\cos(x) = 0$ for $x = \frac{(2k+1)\pi}{2}$, where k is an integer. Now suppose we define $\cot(x) := \frac{1}{\tan(x)}$. We then realise that $\cot(x)$ is the composite function $g(x) := \cot(x) := h(f(x))$, where $h(x) = \frac{1}{x}$ and $f(x) = \tan(x)$. Using our knowledge of composite functions, we see that the domain of $g(x) = \cot(x)$ is the domain of $f(x)$ such that the range of f, $\mathcal{R}(f)$, is contained in the domain of h, $\mathcal{D}(h)$. The range of f is $\mathcal{R}(f) = \mathbb{R}$ and the domain of h is $\mathcal{D}(h) = \mathbb{R}\backslash\{0\}$. Since $\mathbb{R} \not\subseteq \mathbb{R}\backslash\{0\}$, we need to remove all points of the domain of f such that $f(x) = 0$. Indeed, we see that $f(x) = 0$ for all $x = k\pi$, where k is integer. So the domain of $g(x) = \cot(x) = h(f(x))$ is

$$\begin{aligned} \mathcal{D}(f) &= \left\{ x \in \mathbb{R} : x \neq \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z} \right\} \backslash \{x \in \mathbb{R} : x = k\pi,\ k \in \mathbb{Z}\} \\ &= \left\{ x \in \mathbb{R} : x \neq \frac{k\pi}{2},\ k \in \mathbb{Z} \right\}. \end{aligned}$$

So if we define $g(x) = \cot(x)$ as $\frac{1}{\tan(x)}$ we cannot define g at

$$x \in \left\{ ..., -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, ... \right\}.$$

If we look at the sketch of $g(x) = \cot(x)$ above however, $g(x) = 0$ when we have $x = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$; and $g(x) = 0$ seems perfectly reasonable, 0 is a real number. The reason why $g(x) = \cot(x)$ is defined at $x = \frac{(2k+1)\pi}{2}$ is because $\cot(x)$ is not $\frac{1}{\tan(x)}$. The function $\cot(x)$ is defined by $\frac{\cos(x)}{\sin(x)}$, which is equal to $\frac{1}{\tan(x)}$ at all $x \in \left\{ x \in \mathbb{R} : x \neq \frac{k\pi}{2},\ k \in \mathbb{Z} \right\}$, but they are not the same function, since they are defined on different domains. If you want to define $\cot(x)$ as $\frac{1}{\tan(x)}$, you have to restrict to the domain $x \in \left\{ x \in \mathbb{R} : x \neq \frac{k\pi}{2},\ k \in \mathbb{Z} \right\}$ which means that $\cot(x)$ does not intersect the x-axis. Indeed, suppose $\cot(x) = \frac{1}{\tan(x)}$ and consider trying to find the roots of this function. Then

$$\cot(x) = 0 \implies \frac{1}{\tan(x)} = 0 \implies 1 = 0,$$

which is a clear contradiction. Implicitly if we assume that $\cot(x) = \frac{1}{\tan(x)}$ intersects that x-axis, then we are assuming that $\frac{1}{\infty} = 0$, which is not monsterously falacious. The take away from this is that $\cot(x) \neq \frac{1}{\tan(x)}$, but

$$\cot(x) = \frac{\cos(x)}{\sin(x)}.$$

Exercises

Q1. Determine the following exact values

a. $\cos(\frac{\pi}{3})$ b. $\sin(\frac{2\pi}{3})$ c. $\cos(\frac{5\pi}{6})$

d. $\sin(\frac{3\pi}{4})$ e. $\tan(\frac{\pi}{3})$ f. $\tan(\frac{7\pi}{6})$

Q2. Evaluate the following expressions

a. $\sec(\frac{\pi}{4})$ b. $\csc(\frac{5\pi}{3})$ c. $\cot(\frac{5\pi}{6})$

d. $\csc(\frac{\pi}{2})$ e. $\csc(\frac{7\pi}{6})$ f. $\cot(\frac{2\pi}{3})$

g. $\sec(\frac{5\pi}{6})$ h. $\cot(\frac{9\pi}{4})$ i. $\csc(\frac{5\pi}{2})$

Q3. Sketch the following curves, on the domain $x \in [-2\pi, 2\pi]$.

a. $f(x) = 2\cos(x)$.
b. $f(x) = 3\sin(x)$.
c. $f(x) = \frac{1}{2}\cos(2x)$.
d. $f(x) = 2\tan(x)$.
e. $f(x) = \tan(4x)$.
f. $f(x) = \frac{3}{2}\sin(x - \pi)$.
g. $y = \cos(x + \frac{\pi}{3})$.
h. $y = 2\sin(2x - \frac{\pi}{3})$.

Q4. Sketch the following curves on the domain $-\pi \leq x \leq \pi$.

a. $f(x) = 2\sec x$.
b. $f(x) = \frac{1}{3}\csc x$.
c. $f(x) = 2\sec 2x$.
d. $f(x) = \frac{1}{2}\cot x$.
e. $f(x) = \cot 4x$.
f. $f(x) = \frac{2}{3}\csc(x - \pi)$.
g. $f(x) = \sec\left(x + \frac{\pi}{3}\right)$.
h. $f(x) = 2\csc\left(2x - \frac{\pi}{3}\right)$.

Q5. Sketch the following curves, on the domain $x \in [-2\pi, 2\pi]$.

a. $y = |\sec(x)|$ b. $y = |2\cot(x)|$ c. $y = \csc|x|$

Q6. State the domain and range of the following functions.

a. $f(x) = \cos\left(x - \frac{\pi}{3}\right) + 2.$

b. $f(x) = \frac{3}{2}\tan(x - \pi) + 1.$

c. $f(x) = 2\sec(x + \pi).$

d. $f(x) = 3\cot(x) - 5.$

e. $f(x) = \frac{1}{2}\cos\left(x + \frac{\pi}{3}\right) - \frac{3}{\sqrt{5}}.$

f. $f(x) = \frac{1}{3}\csc\left(x + \frac{\pi}{4}\right) - 2.$

Q7. Determine the domain and range of the following functions.

a. $f(x) = \log_e(\sec x).$

b. $f(x) = \tan(e^x).$

c. $f(x) = \cos\left(\frac{1}{\sqrt{x^2-5x+6}}\right).$

d. $f(x) = \csc\left(\frac{1}{x^2+2x+1}\right).$

1.6 Trigonometric Identities

Recall the Pythagorean Identity

$$\sin^2(x) + \cos^2(x) = 1.$$

From this expression, we can derive two more trigonometric identities that can help us in simplifying expressions involving trigonometric functions. Consider

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} &= \frac{1}{\cos^2(x)} \\ \tan^2(x) + 1 &= \sec^2(x). \end{aligned}$$

Therefore, we have derived the trigonometric identity $\tan^2(x) + 1 = \sec^2(x)$ by dividing every term in the Pythagorean identity by $\cos^2(x)$. If we divide every term in the Pythagorean identity by $sin^2(x)$, we can derive another trigonometric identity.

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} &= \frac{1}{\sin^2(x)} \\ 1 + \cot^2(x) &= \csc^2(x). \end{aligned}$$

Therefore the trigonometric identities are

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \tan^2(x) + 1 &= \sec^2(x) \\ 1 + \cot^2(x) &= \csc^2(x). \end{aligned}$$

Example 3.6.1. Simplify the expression

$$\frac{\tan^2(x)}{1 + \tan^2(x)}.$$

Proof. We simply observe that

$$\begin{aligned} \frac{\tan^2(x)}{1 + \tan^2(x)} &= \frac{\tan^2(x)}{\sec^2(x)} \\ &= \frac{\sin^2(x)}{\cos^2(x)} \cdot \frac{1}{\sec^2(x)} \\ &= \frac{\sin^2(x)}{\cos^2(x)} \cdot \cos^2(x) \\ &= \sin^2(x). \end{aligned}$$

□

Exercises

Q1. Show that
$$\sin^4(x) - \cos^4(x) = 1 - 2\cos^2(x).$$

Q2. Show that
$$\tan x \sin x + \cos x = \sec x.$$

Q3. Show that
$$\frac{1}{\tan x} + \tan x = \frac{1}{\sin x \cos x}.$$

Q4. Show that
$$\sin x - \sin x \cos^2 x = \sin^3 x.$$

Q5. Show that
$$\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} = 2\sec x.$$

Q6. Show that
$$\frac{\cos x}{1 - \sin x} - \frac{\cos x}{1 + \sin x} = 2\tan x.$$

Q7. Show that
$$\cos^2 x = \frac{\csc x \cos x}{\tan x + \cot x}.$$

Q8. Show that
$$\frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = 1.$$

Q9. Show that
$$\frac{\tan^2 x}{\tan^2 x + 1} = \sin^2 x.$$

Q10. Show that
$$\frac{1 - \sin x}{\cos x} = \frac{\cos x}{1 + \sin x}.$$

Q11. Show that
$$1 - 2\cos^2 x = \frac{\tan^2 x - 1}{\tan^2 x + 1}.$$

Q12. Show that

$$\tan^2 x = \csc^2 x \tan^2 x - 1.$$

Q13. Show that

$$\sec x + \tan x = \frac{\cos x}{1 - \sin x}.$$

Q14. Show that

$$(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2.$$

Q15. Suppose that $\sec x = -2$ for $\frac{\pi}{2} \leq x \leq \pi$. Determine the exact values for

a. $\cos x$. b. $\tan x$. c. $\cot x$. d. $\csc x$.

Q16. Prove the following identities.

a. $(1 - \tan x)^2 + (1 + \tan x)^2 = 2 \sec^2 x$.

b. $(\cot x - \csc x)^2 = \frac{1+\cos x}{1-\cos x}$.

c. $\sin^2 x \cos^2 y - \cos^2 x \sin^2 y = \sin^2 x - \sin^2 y$.

d. $\sec x + \tan x = \frac{1+\sin x}{\cos x}$.

e. $\sin^2 x \tan x + \cos^2 x \cot x + 2 \sin x \cos x = \tan x + \cot x$.

f. $\sin x(1 + \tan x) + \cos x(1 + \cot x) = \frac{\sin x + \cos x}{\sin x \cos x}$.

1.7 Compound and Double Angle Formulae

The compound angle formulas enable us to calculate the exact values of expressions such as $\sin(\frac{\pi}{12})$ by breaking up $\frac{\pi}{12}$ into $\frac{\pi}{3} - \frac{\pi}{4}$. The double angle formula allow us to do a similar thing, with angles such as $\frac{\pi}{8}$, by expressing $\frac{\pi}{8}$ as $\frac{1}{2} \cdot \frac{\pi}{4}$.

The **compound angle** formulae are listed below:

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \\
\tan(x+y) &= \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \\
\tan(x-y) &= \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}
\end{aligned}$$

The **double angle** formulae are listed below:

$$\begin{aligned}
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 2\cos^2(x) - 1 \\
&= 1 - 2\sin^2(x) \\
\tan(2x) &= \frac{2\tan(x)}{1-\tan^2(x)} \\
&= \frac{\sin(2x)}{\cos(2x)}
\end{aligned}$$

The double angle formula can be derived from the compound angle formulae. For example, if we let $x = x$ and $y = x$. Then we see that

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(x+x) &= \sin(x)\cos(x) + \cos(x)\sin(x) \\
\sin(2x) &= \sin(x)\cos(x) + \sin(x)\cos(x) \\
\sin(2x) &= 2\sin(x)\cos(x).
\end{aligned}$$

Example 3.7.1. Expand the following.

a. $\sin(x - 2y)$.

Proof. We simply observe that

$$\begin{aligned}\sin(x-2y) &= \sin(x)\cos(2y) - \cos(x)\sin(2y) \\ &= \sin(x)(\cos^2(y) - \sin^2(y)) - \cos(x)(2\sin(y)\cos(y)) \\ &= \sin(x)\cos^2(y) - \sin(x)\sin^2(y) - 2\cos(x)\sin(y)\cos(y).\end{aligned}$$

□

b. $\cos(3x)$.

Proof. We simply observe that

$$\begin{aligned}\cos(3x) &= \cos(x+2x) \\ &= \cos(x)\cos(2x) - \sin(x)\sin(2x) \\ &= \cos(x)(1-2\sin^2(x)) - \sin(x)(2\sin(x)\cos(x)) \\ &= \cos(x) - 2\sin^2(x)\cos(x) - 2\sin^2(x)\cos(x) \\ &= \cos(x) - 4\sin^2(x)\cos(x).\end{aligned}$$

□

c. $\tan(2x+y)$.

Proof. Similarly, we have

$$\begin{aligned}\tan(2x+y) &= \frac{\tan(2x)+\tan(y)}{1-\tan(2x)\tan(y)} \\ &= \frac{\frac{2\tan(x)}{1-\tan^2(x)} + \tan(y)}{1 - \frac{2\tan(x)}{1-\tan^2(x)}\tan(y)} \\ &= \frac{2\tan(x) + \tan(y)(1-\tan^2(x))}{1(1-\tan^2(x)) - 2\tan(x)\tan(y)} \\ &= \frac{2\tan(x)+\tan(y)-\tan(x)\tan^2(y)}{1-\tan^2(x)-2\tan(x)\tan(y)}.\end{aligned}$$

□

Example 3.7.2. Find the following exact values using the compound angle formulae.

a. $\cos(\frac{\pi}{12})$.

Proof. Let us simply observe that

$$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \left(\frac{1}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}.\end{aligned}$$

□

b. $\tan(\frac{7\pi}{12})$.

Proof. Similarly, we have

$$\begin{aligned}\tan\left(\frac{7\pi}{12}\right) &= \tan\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\ &= \frac{\tan\left(\frac{\pi}{3}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)}\\ &= \frac{(\sqrt{3})+(1)}{1-(\sqrt{3})\cdot(1)}\\ &= \frac{1+\sqrt{3}}{1-\sqrt{3}}.\end{aligned}$$

□

c. $\csc\left(\frac{5\pi}{12}\right)$.

Proof. It is obvious that

$$\begin{aligned}\csc\left(\frac{5\pi}{12}\right) &= \csc\left(\frac{\pi}{6}+\frac{\pi}{4}\right)\\ &= \frac{1}{\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right)}\\ &= \frac{1}{\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)}\\ &= \frac{1}{(\frac{1}{2})\cdot(\frac{\sqrt{2}}{2})+\frac{\sqrt{3}}{2})\cdot(\frac{\sqrt{2}}{2})}\\ &= \frac{4}{\sqrt{2}+\sqrt{6}}.\end{aligned}$$

□

Example 3.7.3. Find the exact value of $\cos(\frac{\pi}{8})$.

Proof. Using the double angle formula

$$\cos(2x) = 2\cos^2(x) - 1$$

yields

$$\begin{aligned}
\cos\left(\frac{\pi}{4}\right) &= 2\cos^2\left(\frac{\pi}{8}\right) - 1 \\
\frac{\sqrt{2}}{2} + 1 &= 2\cos^2\left(\frac{\pi}{8}\right) \\
\cos^2\left(\frac{\pi}{8}\right) &= \frac{\sqrt{2}+2}{4} \\
\cos\left(\frac{\pi}{8}\right) &= \frac{\pm\sqrt{\sqrt{2}+2}}{2}.
\end{aligned}$$

□

Exercises

Q1. Expand the following, simplifying where possible.

a. $\sin(x - y)$.
b. $\cos(x + 2y)$.
c. $\sin(x + 4y)$.
d. $\cos(2x - \pi)$.
e. $\cos\left(x + \frac{\pi}{3}\right)$.
f. $\sin\left(\frac{1}{2}x - 4y\right)$.

Q2. Determine the exact values of the following.

a. $\cos\left(\frac{\pi}{12}\right)$.
b. $\sin\left(\frac{5\pi}{12}\right)$.
c. $\tan\left(\frac{7\pi}{12}\right)$.
d. $\tan\left(\frac{\pi}{12}\right)$.
e. $\cos\left(\frac{7\pi}{12}\right)$.
f. $\sin\left(\frac{13\pi}{12}\right)$.

Q3. Determine the exact values of the following.

a. $\cos\left(\frac{\pi}{8}\right)$.
b. $\sin\left(\frac{\pi}{8}\right)$.
c. $\tan\left(\frac{\pi}{8}\right)$.

Q4. Suppose that $\cos x = \frac{\sqrt{5}}{4}$. Determine the exact values of the following.

a. $\tan x$.
b. $\sin x$.
c. $\cot x$.
d. $\sin 2x$.
e. $\cos 2x$.
f. $\tan 2x$.
g. $\csc x$.
h. $\csc 2x$.

Q5. Suppose that $\tan x = \frac{1}{\sqrt{5}+1}$. Determine the exact values of the following.

a. $\sec x$.
b. $\sec 2x$.
c. $\sin 3x$.
d. $\tan 2x$.

Q6. Simplify the following expressions.

a. $\frac{3}{5}\sin(x)\cos(-x)$.
b. $\frac{4}{\sqrt{7}}[\sin^2(x) - \cos^2(x)]$.

Q7. Simplify the expression

$$[\cos(-x) + \sin(-x)]^2.$$

Q8. Simplify the following expressions.

a. $\frac{\sin 2x}{1+\cos 2x}$.
b. $\frac{1}{2}\sin 2x \tan x$.
c. $\cos^2 2x - \sin^2 2x$.
d. $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{3}\right)$.
e. $\sin 4x \cos 4x$.
f. $1 + \cos(\pi + 2x)$.

Q9. Simplify the expression

$$\frac{1 - \cos 2x}{1 + \cos 2x}.$$

Q10. Simplify the expression

$$\frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}.$$

Q11. Prove that

$$\frac{\sin 2x \cos x - \cos 2x \sin x}{\cos 2x \cos x + \sin 2x \sin x} = \tan x.$$

Q12. Prove that

$$\frac{\sin x + \cos x \tan y}{\cos x - \sin x \tan y} = \tan(x + y).$$

Q13. Prove that
$$\frac{\tan x - \tan y}{\tan x + \tan y} = \frac{\sin(x-y)}{\sin(x+y)}.$$

Q14. Prove that
$$\cos 4x = 8\cos^4 x - 8\cos^2 x + 1.$$

Q15. Prove that
$$\frac{\cos x + \sin x}{\cos x - \sin x} + \frac{\cos x - \sin x}{\cos x + \sin x} = 2\sec 2x.$$

Q16. Prove that
$$\frac{\sin x + 1 - \cos x}{\sin x - 1 + \cos x} = \frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}.$$

1.8 Inverse Trigonometric Functions

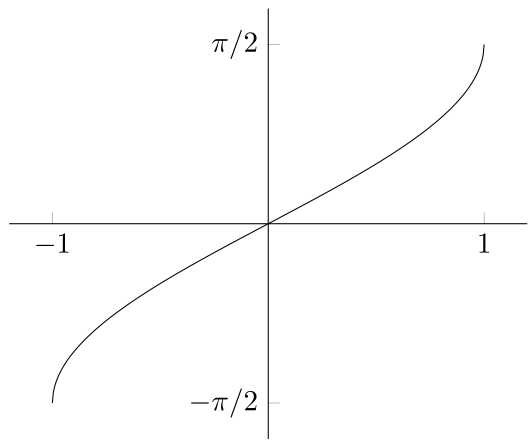
Recall that in chapter one we studied inverse functions. In particular, we saw that for a function to have an inverse, the function had to be injective, or one-to-one. We observe however, that the trigonometric functions, $f(x) = \sin x$, $f(x) = \cos x$ and $f(x) = \tan x$ are clearly not injective. We therefore must restrict the domains on which they are defined. This is done in the following canonical manner.

† We restrict the domain of $f(x) = \sin x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

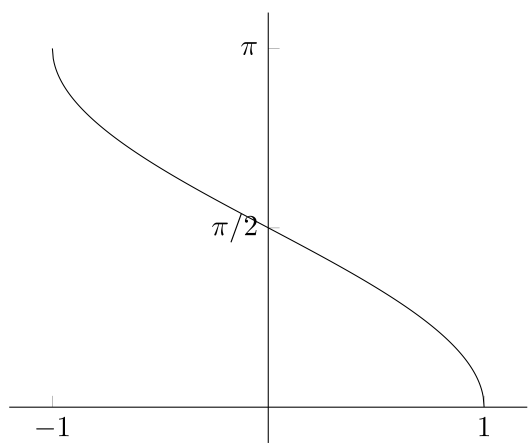
† We restrict the domain of $f(x) = \cos x$ to $0 \leq x \leq \pi$.

† We restrict the domain of $f(x) = \tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

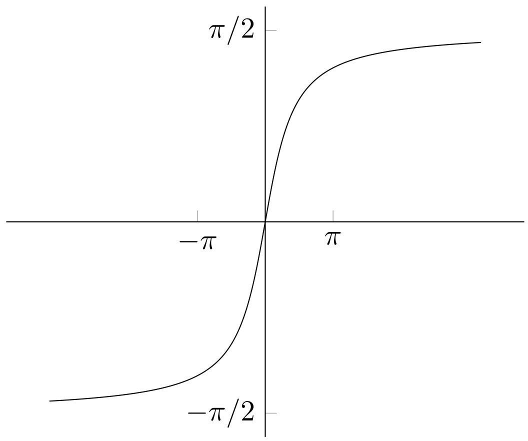
The general shape of the curve $f(x) = \sin^{-1}(x)$ is given by



The general shape of the curve $f(x) = \cos^{-1}(x)$ is given by



The general shape of the curve $f(x) = \tan^{-1}(x)$ is given by



Example 3.8.1. Evaluate the following expressions.

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$.

Proof. We observe that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, and therefore,

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

□

b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$.

Proof. We observe that $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, and therefore,

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

□

Example 3.8.2. Simplify each of the following expressions

a. $\cos^{-1}\left(\cos(\frac{\pi}{6})\right)$.

Proof. We simply observe that

$$\cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

□

b. $\tan\left(\tan^{-1}(\frac{\pi}{3})\right)$.

Proof. We simply observe that

$$\tan\left(\tan^{-1}(\sqrt{3})\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

□

c. $\sin^{-1}\left(\cos(\frac{7\pi}{3})\right)$.

Proof. We simply observe that

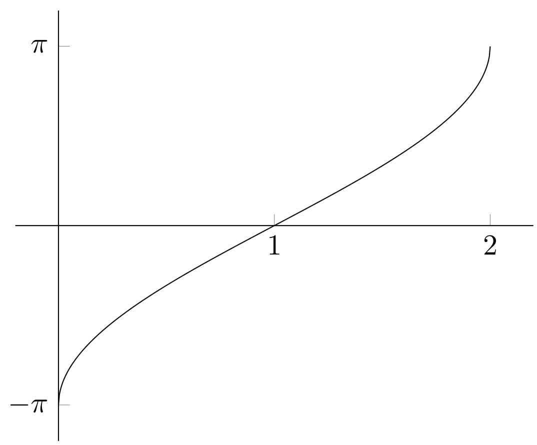
$$\sin^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

□

Example 3.8.3. Sketch the following functions, stating the domain and range.

a. $f(x) = 2\sin^{-1}(x-1)$.

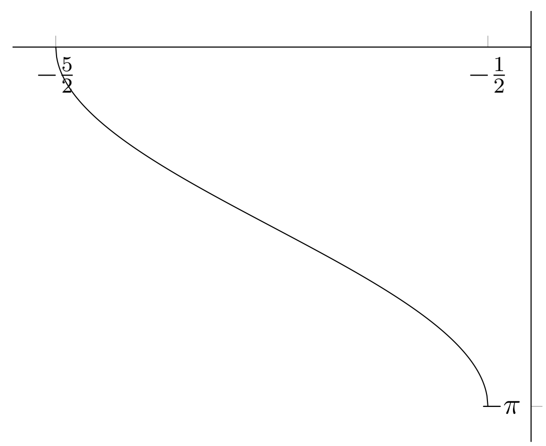
Proof. The function f is obtained from $g(x) := \sin^{-1}(x)$ by simply dilating by factor 2 from the x-axis and translating by 1 unit to the right. Hence we see that



□

b. $f(x) = \cos^{-1}(x + \frac{3}{2}) - \pi$.

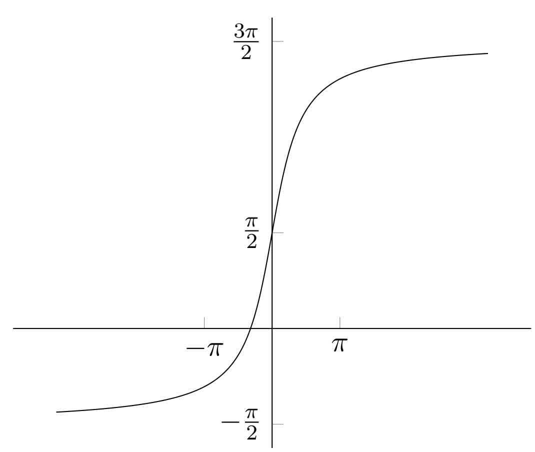
Proof. The function f is obtained from $g(x) := \cos^{-1}(x)$ by simply translating by $\frac{3}{2}$ to the left and by π units down. Hence we see that



∎

c. $f(x) = 2\tan^{-1}(x) + \frac{\pi}{2}$.

Proof. The function f is obtained from $g(x) := \tan^{-1}(x)$ by dilating by factor 2 from the x-axis and translating by $\frac{\pi}{2}$ units down. We therefore see that



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Exercises

Q1 Determine the following exact values

a. $\sin^{-1}(\frac{\sqrt{3}}{2})$. b. $\tan^{-1}(\frac{1}{\sqrt{3}})$. c. $\cos^{-1}(\frac{1}{2})$.

d. $\tan^{-1}(\sqrt{3})$. e. $\cos^{-1}(\frac{\sqrt{2}}{2})$. f. $\sin^{-1}(1)$.

Q2. Determine the following exact values

a. $\sin^{-1}(\sin(\frac{\pi}{3}))$. b. $\cos(\cos^{-1}(\frac{\sqrt{2}}{2}))$. c. $\tan(\sin^{-1}(\frac{\sqrt{3}}{2}))$.

d. $\cos(\tan^{-1}(\sqrt{3}))$. e. $\sin(\cos^{-1}(\frac{1}{2}))$. f. $\tan(\tan^{-1}(1))$.

Q3. Sketch the following inverse circular functions

a. $f(x) = \sin^{-1}(x-1)$.
b. $f(x) = 3\cos^{-1}(x)$.
c. $f(x) = \cos^{-1}(2x)$.
d. $f(x) = \tan^{-1}(x) + \pi$.
e. $f(x) = 2\tan^{-1}(2x)$.
f. $f(x) = \sin^{-1}(2x+2)$.
g. $y = \cos^{-1}(\frac{x}{3}) + 1$.
h. $y = \tan^{-1}(x-2)$.
i. $y = \tan^{-1}(\frac{x}{2}) - \pi$.
j. $y = \sin^{-1}(\frac{x}{3} - 1)$.
k. $y = \cos^{-1}(\frac{x+1}{2})$.
l. $y = \sin^{-1}(\frac{x-2}{3})$.

Q4. Determine the domain and range of the following functions.

a. $f(x) = \sin^{-1}(\sqrt{x})$.
b. $f(x) = \cos^{-1}(\log_e(x-\pi))$.
c. $f(x) = \tan^{-1}(e^x)$.
d. $f(x) = \cos^{-1}\left(\frac{1}{x}\right)$.
e. $f(x) = \sin^{-1}(\sqrt{x^2 - 5x + 6})$.
f. $f(x) = \tan^{-1}|x|$.

1.9 Review Exercises

Q1. Solve the equation

$$\sin(2x) + \frac{\sqrt{3}}{2} = 0,$$

for $0 \leq x \leq 2\pi$.

Q2. Let $f : [-\pi, \pi] \to \mathbb{R}$ be the function defined by

$$f(x) = 1 - 2\cos(3x).$$

Sketch the graph of f, stating all relevant features.

Q3. Prove that

$$\sin x + \cos x = \frac{\tan x}{\sec x} + \frac{\cot x}{\csc x}.$$

Q4. State the domain of

$$\tan^{-1}\left(\frac{1}{\sqrt{x^2 - 9}}\right).$$

Q5. Determine the exact value of

$$\sin\left(\frac{7\pi}{12}\right).$$

Q6. Sketch the graph of

$$f(x) = 3\cot(\pi - x) + 1,$$

stating all relevant features.

Q7. Solve the equation

$$\sin x = \sqrt{3}\cos x,$$

for $0 \leq x \leq 2\pi$.

Q8. Let f be the function defined by

$$f(x) = 3\tan\left(\frac{1}{2}x + \pi\right).$$

Sketch the graph of f on a suitable domain.

Q9. Sketch the graph of

$$f(x) = \cos^{-1}(\pi + 2x) - 3.$$

Q10. Prove that

$$\frac{\tan^2 x \sin^2 x}{\tan^2 x - \sin^2 x} = 1.$$

Q11. Let $f(x) = 1 - 2\sin x$. Describe the transformations necessary to map $f(x)$ to

$$\widetilde{f}(x) = 4 + 3\sin(x - \pi).$$

Q12. Determine the exact value of

$$\cos\left(\frac{\pi}{5}\right).$$

Q13. Solve the equation

$$\cos(2x) = \frac{1}{\sqrt{3}}\sin(2x),$$

for $-\pi \leq x \leq \pi$.

Q14. Let $f(x) = \tan x$. Describe the function $\widetilde{f}(x)$ obtained from applying the following transformations to $f(x)$.

1. Translate by π units to the left.
2. Dilate by factor 2 from the y-axis.
3. Reflect across the x-axis.

Q15. State the domain of

$$f(x) = e^x \tan^{-1}\left(\frac{1}{\sqrt{x^2 - 5x + 6}}\right).$$

Q16. Prove that

$$\frac{\sec x}{\cos x} = 1 + \frac{\tan x}{\cot x}.$$

Q17. Determine the value(s) of $k \in \mathbb{R}$ such that

$$\sin^2\left(\frac{1}{2}x - \pi\right) + 1 = \sin\left(\frac{1}{2}x - \pi\right)$$

has one solution.

Q18. Determine the general solution of

$$2\sin\left(3x + \frac{\pi}{3}\right) - 1 = 0.$$

Q19. Solve the equation

$$\sin x \sec x - 1 = 0,$$

for $-2\pi \leq x \leq 2\pi$.

Q20. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = 1 - 2\tan^{-1}(\pi - x).$$

Sketch the graph of f on a suitable domain, stating all relevant features.

Q21. Prove that

$$\frac{\sec x}{\tan x} - \sin x = \frac{1}{\sec x \tan x}.$$

Q22. Determine the domain of

$$f(x) = \cos^{-1}(\sqrt{x+1}).$$

Q23. Determine the exact value of

$$\tan\left(\frac{5\pi}{12}\right).$$

Q24. Write the general solution to the equaton

$$|2\sin(2x)| = 1.$$

Q25. Solve the equation

$$\sec(2x) = -2,$$

for $0 \leq x \leq 2\pi$.

Q26. Sketch the curve $f : [0, 2\pi] \to \mathbb{R}$ defined by

$$f(x) = 1 - \frac{1}{2}\sin|x + \pi|.$$

Q27. Prove that

$$\frac{\sec x \sin x}{\cot x + \tan x} = \sin^2 x.$$

Q28. Determine the domain of

$$f(x) = \sin^{-1}\left(1 - \frac{3x-2}{4x+7}\right).$$

Q29. Determine the value(s) of $k \in \mathbb{R}$ such that

$$\sin^2(4x) + k\sin(4x) - \frac{3}{k} = 0$$

has two solutions.

Q30. Describe the transformations necessary to map $f(x) = \sin(2x + \pi)$ to

$$\widetilde{f}(x) = \frac{1}{3} - 4\cos(2x).$$