EXAMPLES OF MANIFOLDS WITH NEGATIVE HOLOMORPHIC BISECTIONAL CURVATURE BUT NOT NEGATIVE SECTIONAL CURVATURE

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The holomorphic bisectional curvature was introduced by Goldberg–Kobayashi [3]. If (X, ω) is a Kähler manifold, then

$$\mathrm{HBC}_\omega(u, v) \ := \ \frac{1}{|u|_\omega^2 |v|_\omega^2} R(u, \overline{u}, v, \overline{v}),$$

where $u, v \in T^{1,0}X$. The definition can obviously be extended to Hermitian non-Kähler metrics, but then we have to specify the choice of Hermitian connection. For Kähler metrics, the curvature is always understood (here) to be of the Levi-Civita connection. Moreover, for Kähler metrics, the holomorphic bisectional curvature is the sum of two sectional curvatures.

Example 1. (Wong [2]). Let X be a compact quotient of $\mathbb{B}^3$. Since X admits a Kähler metric of negative bisectional curvature, Kodaira's embedding theorem gives an embedding $\Phi : X \longrightarrow \mathbb{P}^N$. Let H be a hyperplane in $\mathbb{P}^n$ such that $Y := X \cap H$ is smooth. By the subbundle decreasing property, Y supports a Kähler metric of negative bisectional curvature. We will show that Y does not support a Riemannian metric of non-positive sectional curvature, however. The key point is that a complete Riemannian manifold with a Riemannian metric of non-positive sectional curvature is a $K(\Gamma, 1)$–space (Eilenberg–Maclane space). This is an elementary consequence of the Cartan–Hadamard theorem. Observe that by the Lefschetz hyperplane section theorem, $\pi_1(X) \simeq \pi_1(Y)$. Therefore,

$$\mathbb{Z} \simeq H_6(X, \mathbb{Z}) \simeq H_6(\pi_1(X), \mathbb{Z}) \simeq H_6(\pi_1(Y), \mathbb{Z}) \not\simeq H_6(Y, \mathbb{Z}) = 0,$$

which implies that X cannot be an Eilenberg–Maclane space.

The following example due to Mohsen [4] shows that a complete simply connected Kähler manifold with negative holomorphic bisectional curvature can be compact. Of course, by Cartan–Hadamard, this is not possible if the holomorphic bisectional curvature is replaced by the sectional curvature.

Theorem 2. (Mohsen [4]). There are complete intersections of dimension d defined by equations of degree k in $\mathbf{P}^n$ with Kähler metrics of negative holomorphic bisectional curvature if $n \geq 4d - 1$.

Mohsen's theorem is an analytic brother of the work of Brotbek–Darondeau [1] and Xie [5] who constructed complete intersections with ample cotangent bundles.

Remark. To the author's knowledge, the only known way of distinguishing between the holomorphic bisectional curvature and the sectional curvature is via the Cartan–Hadamard theorem.

References

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