THE $\mathcal{C}^2$–ESTIMATE FOR THE COMPLEX MONGE–AMPÈRE EQUATION

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Let (X, ω) be a compact Kähler manifold. We adopt the notation $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for a cohomologous Kähler metric. We assume that there is a function $f : X \to \mathbf{R}$ such that

$$\omega_\varphi^n \quad = \quad e^f \omega^n. \tag{0.1}$$

In this talk, we will discuss the following $\mathcal{C}^2$–estimate due to Aubin [**?**] and Yau [**?**]:

Theorem. Let $\varphi \in \mathcal{C}^4(X)$ be a solution to the complex Monge–Ampère equation (0.1). There is a constant $C = C(X, \|f^{1/(n-1)}\|_{W^{2,\infty}}) > 0$ such that

$$\sup_X |\Delta_\omega \varphi| \quad \leq \quad C,$$

where $\Delta_\omega \varphi := \text{tr}_\omega(\sqrt{-1}\partial\bar{\partial}\varphi)$ is the *complex Laplacian* of the Kähler metric ω.

The lower bound on $\Delta_\omega\varphi$ is automatic, depending only on the dimension of X. Indeed, taking the trace of $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ yields $\text{tr}_\omega(\omega_\varphi) \;=\; n + \Delta_\omega\varphi \;>\; 0$, i.e.,

$$\Delta_\omega \varphi > -n.$$

Moreover, an upper bound on $\Delta_\omega\varphi$ is equivalent to an upper bound on $\text{tr}_\omega(\omega_\varphi)$.

The method of estimating $\text{tr}_\omega(\omega_\varphi)$ goes back to Pogorelov's work [**?**] on $\mathcal{C}^2$ estimates for the real Monge–Ampère equation on bounded convex domains. In more detail, we will construct an auxilliary, globally defined, function $\mathcal{Q} = \mathcal{Q}(\text{tr}_\omega(\omega_\varphi))$ on X. Since X is compact, there is a point $x_0 \in X$ for which $\mathcal{Q}$ attains a maximum. At this point, the gradient vanishes and $0 \geq \Delta_\omega \mathcal{Q}$. Hence, it suffices to obtain constants $C_1 \geq 0$ and $C_2 > 0$ such that

$$\Delta_\omega \mathcal{Q} \quad \geq \quad -C_1 + C_2 \text{tr}_{\omega_\varphi}(\omega). \tag{0.2}$$

At the point $x_0 \in X$ where $\mathcal{Q}$ achieves its maximum, we would then have

$$0 \quad \geq \quad \Delta_\omega \mathcal{Q} \;\geq\; -C_1 + C_2 \text{tr}_{\omega_\varphi}(\omega),$$

and thus $\text{tr}_\omega(\omega_\varphi) \leq C_1/C_2$.

Remark. Although we are yet to prove anything yet, some remarks are in order. The first is that we are not obligated to use the Laplacian of ω. We can equally well use the Laplacian of ω_{φ} (and in fact, this is what Aubin and Yau used). The difference in Laplacians emerges impacts what geometric properties of X the constant C in the main estimate depends on, however.

The second important remark is that in the choice of auxilliary function $\mathcal{Q}$, the bound on $\text{tr}_{\omega}(\omega_{\varphi})$ coming from (0.2) occurs at the point where $\mathcal{Q}$ attains its maximum. Hence, one needs to ensure that this is also the point where $\text{tr}_{\omega}(\omega_{\varphi})$ attains its maximum if one wants to obtain the estimate $\text{tr}_{\omega}(\omega_{\varphi}) \leq C_1/C_2$ globally on X.

Regardless of the specific choice of auxiliary function $\mathcal{Q}$, to compute $\Delta_{\omega}\mathcal{Q}$, we will need to compute $\Delta_{\omega}\text{tr}_{\omega}(\omega_{\varphi})$. This has typically been done in a direct ad-hoc manner, but there is a more high-brow perspective that becomes essential in certain contexts (see, e.g., [**?**, **?**, **?**]).

There is some complex function theory lurking in the background of these calculations. Indeed, if $f : (X, \omega_g) \to (Y, \omega_h)$ is a holomorphic map between Kähler manifolds, we may locally write $w^{\alpha} = f^{\alpha}(z_1, ..., z_n)$, where $w = (w^1, ..., w^m)$. The derivative ∂f is then locally described by the $(n \times m)$ matrix $\partial f = (f_k^{\alpha}) := \frac{\partial f^{\alpha}}{\partial z_k}$. The norm (squared) of ∂f is then

$$|\partial f|^2 \ = \ g^{k\bar{\ell}} f_k^{\gamma} \overline{f_{\ell}^{\delta}} h_{\gamma\bar{\delta}}, \tag{0.3}$$

where locally $\omega_g = \frac{\sqrt{-1}}{2} g_{k\bar{\ell}} dz^k \wedge d\bar{z}^{\ell}$ and $\omega_h = \frac{\sqrt{-1}}{2} h_{\gamma\bar{\delta}} dw^{\gamma} \wedge d\bar{w}^{\delta}$. If we take $f = \text{id}$ to be the identity map, then $f_k^{\gamma} = \delta_k^{\gamma}$, and (0.3) reads

$$|\partial \text{id}|^2 \ = \ g^{k\bar{\ell}} \delta_k^{\gamma} \delta_{\ell}^{\delta} h_{\gamma\bar{\delta}} \ = \ g^{k\bar{\ell}} h_{k\bar{\ell}} \ = \ \text{tr}_{\omega_g}(\omega_h).$$

Hence, $\mathcal{C}^2$–estimates are obtained from obtaining estimates on the energy density $|\partial f|^2$ of a holomorphic map.