Exponentials and Logarithms - Practice Exam 2

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Question 1. Solve the following equation for $x \in \mathbb{R}$,

$$\log_e(x) + \log_e(x-3) = 4.$$

Proof. We simply observe that

$$\begin{aligned} \log_e(x) + \log_e(x-3) = 4 \quad &\implies \quad \log_e[x(x-3)] = 4 \\ &\implies \quad x(x-3) = e^4 \\ &\implies \quad x^2 - 3x - e^4 = 0 \\ &\implies \quad x = \frac{3 \pm \sqrt{9+4e^4}}{2}. \end{aligned}$$

□

Question 2. State the transformations (in order) that are required to transform the function

$$f(x) = \frac{1}{3}e^{2x+7} - 6$$

to the function

$$\widetilde{f}(x) = e^x.$$

Proof. Dilating by factor 3 from the x-axis maps $f(x) = \frac{1}{3}e^{2x+7} - 6$ to $f_1(x) = e^{2x+7} - 18$. Dilating by factor 2 from the y-axis maps $f_1(x)$ to $f_2(x) = e^{x+\frac{7}{2}} - 18$. Translating by 18 units up maps $f_2(x)$ to $f_3(x) = e^{x+\frac{7}{2}}$. Translating by $\frac{7}{2}$ units to the right maps $f_3(x)$ to $\widetilde{f}(x) = e^x$. □

Question 3. Sketch the functions f and $\widetilde{f}$ that are given in Question 2.

Proof. The x-intercepts of $f(x)$ occur when $f(x) = 0$. Therefore, we see that

$$\begin{aligned} \frac{1}{3}e^{2x+7} - 6 = 0 \quad &\implies \quad e^{2x+7} = 18 \\ &\implies \quad 2x + 7 = \log_e(18) \\ &\implies \quad x = \frac{1}{2}\left(\log_e(18) - 7\right). \end{aligned}$$

The y intercept of $f(x)$ is given by

$$y = \frac{1}{3}e^7 - 6.$$

There is no x-intercept for $\widetilde{f}(x)$ and the y-intercepts for $\widetilde{f}(x)$ is simply $y = 1$. See an online graphing calculator for the shapes of the curves. □

Question 4. Determine the value of $k \in \mathbb{R}$ such that the equation

$$\log_4(x) - k\log_x(4) = 3$$

has two solutions.

Proof. Begin by converting all logarithms to the same base. That is, write

$$\log_4(x) - k\log_x(4) = 3 \implies \log_4(x) - k\frac{1}{\log_4(x)} = 3.$$

Now let $u = \log_4(x)$ and observe that

$$\begin{aligned}\log_4(x) - k\frac{1}{\log_4(x)} = 3 &\implies u - \frac{k}{u} = 3\\ &\implies u^2 - 3u - k = 0.\end{aligned}$$

The descriminant is given by $\Delta = b^2 - 4ac > 0$ for two solutions. Hence, we see that

$$\begin{aligned}\Delta = b^2 - 4ac > 0 &\implies 9 + 4k > 0\\ &\implies k > -\frac{9}{4}.\end{aligned}$$

□