New Vanishing Results for Kähler Manifolds

Peter Petersen

NZMS-AustMS-AMS Meeting 2024

November 21, 2024

Let (M^n, g) be a compact connected Riemannian n-manifold and ϕ a k-form on M.

The Weitzenböck Formula:

$$\Delta\phi = (dd^* + d^*d)\,\phi = \nabla^*\nabla\phi + \mathrm{Ric}_L(\phi),$$

here $\mathrm{Ric}_L(\phi)$ is a contraction of $\nabla^2\phi$ which as Weitzenböck realized had to be a contraction of $R \otimes \phi$ as it is a 0th order invariant.

Bochner-Yano: When the form is harmonic, $\Delta\phi = 0$, we obtain

$$0 = g(\nabla^*\nabla\phi, \phi) + g(\mathrm{Ric}_L(\phi), \phi).$$

Here

$$\int g\left(\nabla^*\nabla\phi,\phi\right)=\int|\nabla\phi|^2\geq 0.$$

In particular,

$$g\left(\mathrm{Ric}_L\left(\phi\right),\phi\right)\geq 0\Rightarrow|\nabla\phi|^2=0,\ g\left(\mathrm{Ric}_L\left(\phi\right),\phi\right)=0.$$

When $g\left(\mathrm{Ric}_L\left(\phi\right),\phi\right)\geq 0$ on all k-forms Hodge theory shows

$$b_k=\dim H^k\left(M\right)\leq\binom{n}{k}=\dim H^k\left(S^1\times\cdots\times S^1\right)$$

and if in addition $g\left(\mathrm{Ric}_L\left(\phi\right),\phi\right)>0$ at a point then

$$b_k=0.$$

This strategy carries over to Kähler manifolds (X^n, g) where X is a complex manifold of complex dimension n. Here k-forms on the complexified tangent bundle $T^{\mathbb{C}}M$ are further divided into (p,q)-forms, $p+q=k$, that look like

$$\phi = \sum \phi_{I\bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}.$$

The corresponding cohomology groups are denoted by $H^{p,q}(X)$. Complex conjugation shows that (p,q)-forms are conjugate isomorphic to (q,p)-forms. So it suffices to consider real forms in $H^{p,q} \oplus H^{q,p}$ if we wish to control Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$.

The techniques involved in our results depend on the basic Lie algebra actions on tensors derived from the regular representation. For an endomorphism $S \in \mathfrak{gl}(T_x M)$ we have

$$(S\omega)(v_1, \ldots, v_p) = -\sum_{i=1}^{p} \omega(v_1, \ldots, Sv_i, \ldots, v_p).$$

With this notation we can in the Riemannian case select an ONB $A_\alpha \in \Lambda^2 T_x M$ of eigenvectors $R(A_\alpha) = \lambda_\alpha A_\alpha$ and the curvature term in the Bochner formula becomes:

$$g(\mathrm{Ric}_L(\phi), \phi) = \sum \lambda_\alpha |A_\alpha \phi|^2.$$

This formula is due to Poor and leads immediately to a proof of the Gallot-Meyer classification of manifolds with nonnegative curvature operator.

P-Wink used this formula to obtain restrictions on Betti and Hodge numbers with less restrictive curvature assumptions. For Kähler manifolds we consider the Kähler curvature operator

$$K : \Lambda^{1,1} T^{\mathbb{C}} M \to \Lambda^{1,1} T^{\mathbb{C}} M$$

and consider an ONB of eigenvalues $K(A_\alpha) = \lambda_\alpha A_\alpha$ and obtain a similar formula

$$g\left(\mathrm{Ric}_L(\phi), \bar{\phi}\right) = \sum \lambda_\alpha |A_\alpha \phi|^2 .$$

We say that a self-adjoint operator is r-positive provided its eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ satisfy

$$\lambda_1 + \cdots + \lambda_{\lfloor r \rfloor} + (r - \lfloor r \rfloor) \lambda_{\lfloor r \rfloor + 1} > 0.$$

Bochner 1946: If Ric is $2p$-positive for $p = 1, ..., n$, then $h^{p,0} = 0$.

Ogiue-Tachibana: 1978: If K is positive, then $h^{p,q} = 0$ for $p \neq q$ and $h^{p,p} = 1$, i.e., X^n has the cohomology of $\mathbb{P}^n$.

P-Wink 2021: If K is $2 + 2/n$ positive, then X^n has the cohomology of $\mathbb{P}^n$.

This result is not ideal and in fact is much better for (p, p)-forms requiring only $n/2$-positivity. It is possible that this is due to the fact that we only use that K is self-adjoint thus ignoring the Bianchi identity.

For Kähler manifolds the symmetries of the curvature tensor $R_{a\bar{b}c\bar{d}}$ however allow for a more efficient self-adjoint operator that does include the Bianchi identity. The key identities used for K are

$$R_{a\bar{b}c\bar{d}} = R_{c\bar{d}a\bar{b}} = -R_{\bar{b}ac\bar{d}}.$$

For Kähler curvature tensors the Bianchi identity is encoded in

$$R_{a\bar{b}c\bar{d}} = R_{c\bar{b}a\bar{d}} = R_{a\bar{d}c\bar{b}}.$$

This also tells us that the curvature tensor is a self-adjoint operator on the space of holomorphic symmetric tensors $S^{2,0}$ spanned by

$$\partial_a \odot \partial_b = \partial_a \otimes \partial_b + \partial_b \otimes \partial_a.$$

This operator $C : S^{2,0} \to S^{2,0}$ was introduced by Calabi-Vesentini in an important paper (1960) that initiated the study of rigidity of locally symmetric spaces. It seems only reasonable to refer to it as the Calabi curvature operator.

C-V observed that any self-adjoint operator on $S^{2,0}$ is in fact an algebraic Kähler curvature tensor. They also calculated the eigenvalues of C on the irreducible Hermitian symmetric spaces. In constant holomorphic curvature it is a homothety. For higher rank irreducible symmetric spaces C has precisely two eigenvalues of opposite sign. Among these examples the complex quadric

$$\frac{SO(2+n)}{SO(2)\times SO(n)}$$

is the most positive with the eigenvalues satisfying

$$\sigma_1+\cdots\sigma_{\lfloor \frac{n}{2}\rfloor}+\left(\frac{n}{2}-\left\lfloor\frac{n}{2}\right\rfloor\right)\sigma_{1+\lfloor\frac{n}{2}\rfloor}=0.$$

Ogiue-Tachibana 1978: If C is positive, then the space has the cohomology of $\mathbb{P}^n$.

Broder-Nienhaus-P-Stanfield-Wink: If C is $\frac{n}{2}$-positive, then X^n has the cohomology of $\mathbb{P}^n$.

Moreover, if C is $n/2$-nonnegative, then one of the following cases holds:

- The holonomy is irreducible and the space either has the cohomology of $\mathbb{P}^n$ or is isometric to the complex quadric.
- The holonomy is reducible and a finite cover is isometric to $T^k \times Y$, where Y is a product of spaces that are biholomorphic to projective spaces.

In the first case it follows from Berger's classification of holonomy groups that the holonomy is either $U(n)$ or the space is symmetric as it can't be Ricci flat. In the second case the reducibility introduces so many zero eigenvalues that C is forced to be nonnegative. In particular, the bisectional curvature is nonnegative and we can use Mok's classification (1988).

Consider an ONB of eigenvalues $C(S_\alpha) = \sigma_\alpha S_\alpha$. The holomorphic tensors S_α can be type changed to conjugate linear maps $T^{1,0} \to T^{0,1}$. As such they act on tensors and in particular (p,q)-forms, however $S_\alpha : \Lambda^{p,q} \to \Lambda^{p+1,q-1}$.

We have

$$g\left(\mathrm{Ric}_L(\phi), \bar{\phi}\right) = 8 \sum \sigma_\alpha |S_\alpha \phi|^2 .$$

If $|\phi|^2 = 1$, then

$$\sum |S_\alpha \phi|^2 = \frac{1}{4}\left((n+1)(p+q) - 2pq\right)$$

and when ϕ is a primitive real form in $\Lambda^{p,q} \oplus \Lambda^{q,p}$, then

$$|S_\alpha \phi|^2 \leq \frac{1}{2} + \min\left\{p, q, \frac{\sqrt{pq}}{2}\right\}$$

If we define

$$C_{p,q} = \frac{(n+1)(p+q) - 2pq}{2 + \min\left\{4p, 4q, 2\sqrt{pq}\right\}},$$

then real primitive harmonic forms in $\Lambda^{p,q} \oplus \Lambda^{q,p}$ vanish provided C is $C_{p,q}$-positive.

Here:

- $C_{p,q} \geq \frac{n}{2}$ and the minimum is attained when $p = q = 1$.
- $C_{p,p} = \frac{(n+1)p - p^2}{1+p}$.
- $C_{p,0} = \frac{(n+1)p}{2}$ and in particular $C_{n,0} = \dim S^{2,0}$.