LAURET'S QUESTION ON GAUDUCHON RICCI-FLAT METRICS

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Let ${}^c\mathrm{Ric}(\omega) :=_{\mathrm{loc}} -\sqrt{-1}\,\partial\bar{\partial}\log\omega^n$ denote the first Chern Ricci form of a Hermitian metric ω. For $t \in \mathbf{R}$, the first t-Gauduchon Ricci form is defined by

$$^t\mathrm{Ric}(\omega) \ := \ {}^c\mathrm{Ric}(\omega) + \frac{t-1}{2}\big(\partial\partial^\star\omega + \bar{\partial}\bar{\partial}^\star\omega\big).$$

Any left-invariant Hermitian metric on a compact complex parallelizable manifold is balanced and Chern-flat (see, e.g., [Fin25]), and hence satisfies ${}^t\mathrm{Ric}(\omega) = 0$ for every $t \in \mathbf{R}$. In the non-parallelizable homogeneous setting, Jorge Lauret has shown that ${}^t\mathrm{Ric}(\omega) = 0$ forces $t < 1$. In an email (April 11, 2026), he asked me whether there exist compact examples that are non-Kähler, non-Chern-flat, and satisfy ${}^t\mathrm{Ric}(\omega) = 0$ for some $t > 1$.

Theorem 1.1. Let (Y^m, ω_Y) be a compact balanced manifold with ${}^c\mathrm{Ric}(\omega_Y) = 0$. Let $E = \mathbf{C}/\mathbf{Z}[\sqrt{-1}]$ be an elliptic curve. Fix a smooth non-constant 1-periodic function $f : \mathbf{R} \to \mathbf{R}$ in the variable $x := \mathrm{Re}(z)$, where z is the holomorphic coordinate on the universal cover of E. For any $t \in \mathbf{R}$, define a Hermitian metric $\omega_{t,f}$ on $X = E \times Y$ by

$$\omega_{t,f} \quad := \quad \frac{\sqrt{-1}}{2}e^{-tmf(x)}\mathrm{d}z \wedge \mathrm{d}\bar{z} + e^{f(x)}\omega_Y.$$

Then ${}^t\mathrm{Ric}(\omega_{t,f}) = 0$.

Proof. It will be convenient to write g for the metric whose $(1,1)$-form is $\omega_{t,f}$. First observe that the volume forms are related by $\omega_{t,f}^{m+1} = \frac{\sqrt{-1}}{2}(m+1)e^{m(1-t)f(x)}\mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \omega_Y^m$. Since ${}^c\mathrm{Ric}(\omega_Y) = 0$, we have

$$^c\mathrm{Ric}(\omega_{t,f}) \quad = \quad m(t-1)\sqrt{-1}\partial\bar{\partial}f.$$

Since $\omega_{t,f}$ is a warped product, the Christoffel symbols for the Chern connection are readily computable. Indeed, let $(z, w^1, ..., w^m)$ denote the local holomorphic coordinates on X, with 0 indicating the E coordinate, and Greek indices denoting the Y coordinates. In particular, we write $\omega_Y = \sqrt{-1}h_{\alpha\bar{\beta}}(w, \bar{w})\mathrm{d}w^\alpha \wedge \mathrm{d}\bar{w}^\beta$. The Hermitian matrix is block diagonal: $\mathrm{g}^{00} = \frac{1}{2}e^{-tmf(x)}$, $\mathrm{g}_{\alpha\bar{\beta}} = e^{f(x)}h_{\alpha\bar{\beta}}$, and $\mathrm{g}_{0\bar{\beta}} = \mathrm{g}_{\alpha\bar{0}} = 0$. Hence, $\mathrm{g}^{0\bar{0}} = 2e^{tmf(x)}$, $\mathrm{g}^{\alpha\bar{\beta}} = e^{-f(x)}h^{\alpha\bar{\beta}}$. Then, with $\partial f := \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - \sqrt{-1}\frac{\partial f}{\partial y}) = \frac{1}{2}\frac{\partial f}{\partial x}$,

$$\Gamma_{00}^0 = -tm\partial f, \qquad \Gamma_{0\beta}^\alpha = \partial f\delta_\beta^\alpha, \qquad \Gamma_{\beta\gamma}^\alpha = h^{\alpha\bar{\delta}}\partial_\beta h_{\gamma\bar{\delta}},$$

with all other Chern Christoffel symbols vanishing. Since ω_Y is balanced, and $\tau = \sum_i {}^c\mathrm{T}_{ij}^i$ denotes the torsion $(1,0)$-form for $\omega_{t,f}$, then

$$\tau_0 \ = \ \sum_{\alpha=1}^m {}^c\mathrm{T}_{\alpha 0}^{\alpha} \ = \ -m\partial f, \qquad \tau_\beta \ = \ \tau_\beta(\omega_Y) \ = \ 0.$$

Hence, $\tau = -m\partial f$, and $\omega_{t,f}$ is balanced if and only if f is constant. Therefore, with $\bar{\partial}^\star \omega_{t,f} = -\sqrt{-1}\tau$, it follows that

$$\partial\partial^\star \omega_{t,f} + \bar{\partial}\bar{\partial}^\star \omega_{t,f} \ = \ -2m\sqrt{-1}\partial\bar{\partial} f.$$

Therefore,

$${}^t\mathrm{Ric}(\omega_{t,f}) \ = \ m(t-1)\sqrt{-1}\partial\bar{\partial} f + \frac{t-1}{2}(-2m\sqrt{-1}\partial\bar{\partial} f) \ = \ 0.$$

□

Remark 1.2. Theorem 1.1 applies to any compact Kähler manifold Y with (holomorphically) torsion canonical bundle. In particular, complex tori, K3 or Enriques surfaces, smooth hypersurfaces in $\mathbf{P}^n$ of degree $n+1$, hyper-Kähler manifolds. The metrics $\omega_{t,f}$ are not Kähler if f is non-constant, however. It is worth comparing this behavior with that of the second Chern Ricci-flat metrics on tori [BP25].

Example 1.3. We give some non-Kähler examples.

(1) The Iwasawa threefold with its standard metric. More generally, any left-invariant Hermitian metric on a compact complex parallelizable manifold is balanced and Chern-flat.
(2) The Goldstein–Prokushkin torus bundles over an algebraic K3 surface [GP04] are non-Kähler threefolds with balanced first Chern Ricci-flat metrics (see [GS24, Example 2.5]).
(3) All small resolutions of smoothable projective nodal Kähler Calabi–Yau threefolds admit Chern-Ricci-flat balanced metrics [GS25]. More generally, crepant resolutions of compact orbifolds carrying singular Chern Ricci-flat balanced metrics admit smooth Chern Ricci-flat balanced metrics [FG25, GS24].

Remark 1.4. Several examples can be generated once the Calabi–Yau problem in the balanced class (see [Tos15, Conjecture 4.1]) is resolved. For example, let $Y_k := \sharp_k(\mathbf{S}^3 \times \mathbf{S}^3)$ with its Lu–Tian complex structure. Then $K_{Y_k} \simeq \mathcal{O}_{Y_k}$, and from [FLY12], there are balanced metrics on Y_k. To my knowledge, the conjecture remains open, and even in the special case of Y_k, I am not familiar with any results demonstrating that balanced first Chern Ricci-flat metrics exist on Y_k.

References

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