Hyperbolicity Might Nevertheless Be Algebraic*

Kyle Broder
The University of Queensland
MATRIX – Geometry with Symmetries

Joint work with Frédéric Campana (University of Lorraine).
Remarks on Lang's Conjectural Characterisation of Hyperbolicity of Projective Manifolds, arXiv:2410.06402.

* Title plagiarised from the famous article written by Miles Reid.

Main Results of the Talk

A compact complex manifold X is (Brody) hyperbolic if every holomorphic map $\mathbf{C} \to X$ is constant (no entire curves).

Obstructions to hyperbolicity: rational curves $\mathbf{P}^1 \to X$ and images of abelian varieties $\mathbf{T}^n \to X$. S. Lang (1986) conjectured that for projective manifolds these are the only obstructions.

Theorem 1. (B.–Campana 2024). Lang's conjecture follows from the abundance conjecture and the existence of rational curves on (terminal) Calabi–Yau and hyper-Kähler varieties.

It is not sufficient to test hyperbolicity on algebraic curves:

Theorem 2. (B.–Campana 2024). There are projective manifolds with a Kähler–Einstein metric $\mathrm{Ric}(\omega) = -\omega$, no rational curves $\mathbf{P}^1 \to X$, no elliptic curves $\mathbf{T}^1 \to X$, but admits (Zariski dense) entire curve $\mathbf{C} \to X$.

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Proliferation mechanisms. (1) Complex submanifolds, (2) products, (3) deformations, and (4) universal coverings ($\Longleftrightarrow$).

Obstructions. Rational curves $\mathbf{P}^1 \to X$, elliptic curves $\mathbf{T} \to X$. More generally, holomorphic maps from $\mathbf{P}^n$ and abelian varieties (complex tori) $\mathbf{T}^n$.

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The Holomorphic Sectional Curvature

A sign on the holomorphic sectional curvature

$$\mathrm{HSC}_\omega(\xi) \ := \ \frac{1}{|\xi|^4}\mathrm{R}(\xi,\bar{\xi},\xi,\bar{\xi}), \qquad \xi \in \mathcal{T}_X^{1,0}$$

forces a sign on the scalar curvature, e.g., $\mathrm{HSC}(\omega) < 0 \implies \mathrm{Scal}(\omega) < 0$ if the metric is Kähler.

It does not force a sign on the Ricci curvature: There are an infinite number of $\mathbf{P}^1$-bundles over $\mathbf{P}^1$ with $\mathrm{HSC}(\omega) > 0$ but no Kähler metrics of positive Ricci curvature.

A sign on the Ricci curvature also does not force a sign on the holomorphic sectional curvature: A smooth hypersurface in $\mathbf{P}^n$ of degree $d > n+1$ admits a Kähler–Einstein metric with $\mathrm{Ric}(\omega) = -\omega$. The Fermat hypersurface $X_d := \{z_0^d + \cdots + z_n^d = 0\} \subseteq \mathbf{P}^n$ has rational curves $\mathbf{P}^1 \to X_d$ and thus, cannot have a metric with $\mathrm{HSC}(\omega) < 0$.

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The Wu–Yau Theorem

The examples of $\mathbf{P}^1$-bundles over $\mathbf{P}^1$ with $\mathrm{HSC}(\omega) > 0$ but no metrics of positive Ricci curvature, makes the following rather striking.

Theorem. (Wu–Yau, Tosatti–Yang). A compact Kähler manifold (X, ω) with $\mathrm{HSC}(\omega) < 0$ admits a Kähler–Einstein metric $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ with $\mathrm{Ric}(\omega_\varphi) = -\omega_\varphi$.

Main techniques: PDE methods (complex Monge–Ampère) and Schwarz lemma ($\mathcal{C}^2$-estimate).

These results provide significant confirmation of the following long-standing conjecture of Kobayashi (1970) and Lang (1986).

Conjecture. (Kobayashi, Lang). A compact hyperbolic Kähler manifold admits a Kähler–Einstein metric with negative Ricci curvature.

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The existence of a Kähler–Einstein metric $\mathrm{Ric}(\omega) = -\omega$ is equivalent to (some tensor multiple of) the canonical bundle $K_X = \det(\mathcal{T}_X^\star)$ having sufficiently many sections (i.e., K_X is ample).

For a curve C, the genus is given by $g(C) = \dim_{\mathbf{C}} H^0(C, K_C)$. In particular, if C is negatively curved, $g(C) \geq 2$, and $\dim_{\mathbf{C}} H^0(C, K_C) \geq 2$.

Definition. The Kodaira dimension of X is

$$\kappa_X \quad := \quad \limsup_{\ell \to \infty} \frac{\log \dim_{\mathbf{C}} H^0(X, K_X^{\otimes \ell})}{\log(\ell)}.$$

If $H^0(X, K_X^{\otimes \ell}) = \{0\}$ for all $\ell \in \mathbf{N}$, we say that $\kappa_X = -\infty$.

Facts. (1) $\kappa_X \in \{-\infty, 0, 1, ..., \dim_{\mathbf{C}} X\}$, (2) Kähler–Einstein with $\mathrm{Ric}(\omega) = -\omega$ have maximal Kodaira dimension $\kappa_X = \dim_{\mathbf{C}} X$ (general type), (3) Ricci-flat Kähler $\mathrm{Ric}(\omega) = 0 \implies \kappa_X = 0$. In particular, tori $\mathbf{T}^n$, K3 surfaces, Calabi–Yau and hyper-Kähler manifolds have $\kappa_X = 0$.

Kobayashi–Lang Conjecture Without a Negatively Curved Metric.

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For a curve C, the genus is given by $g(C) = \dim_{\mathbf{C}} H^0(C, K_C)$. In particular, if C is negatively curved, $g(C) \geq 2$, and $\dim_{\mathbf{C}} H^0(C, K_C) \geq 2$.

Definition. The Kodaira dimension of X is

$$\kappa_X \quad := \quad \limsup_{\ell \to \infty} \frac{\log \dim_{\mathbf{C}} H^0(X, K_X^{\otimes \ell})}{\log(\ell)}.$$

If $H^0(X, K_X^{\otimes \ell}) = \{0\}$ for all $\ell \in \mathbf{N}$, we say that $\kappa_X = -\infty$.

Facts. (1) $\kappa_X \in \{-\infty, 0, 1, ..., \dim_{\mathbf{C}} X\}$, (2) Kähler–Einstein with $\mathrm{Ric}(\omega) = -\omega$ have maximal Kodaira dimension $\kappa_X = \dim_{\mathbf{C}} X$ (general type), (3) Ricci-flat Kähler $\mathrm{Ric}(\omega) = 0 \implies \kappa_X = 0$. In particular, tori $\mathbf{T}^n$, K3 surfaces, Calabi–Yau and hyper-Kähler manifolds have $\kappa_X = 0$.

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Conjecture. (Kobayashi, Lang). A compact hyperbolic Kähler surface admits a Kähler–Einstein metric with negative Ricci curvature. In particular, $\kappa_X = \dim_{\mathbf{C}} X = 2$.

Proof: If X is hyperbolic, then X has no rational curves $\mathbf{P}^1 \to X$. Since a Kähler surface has $\kappa_X = -\infty$ if and only if it is covered by rational curves (uniruled), we have $\kappa_X \geq 0$. We then consider each case:

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Mori's Theorem and Abundance

Theorem. (Mori). A projective manifold with no rational curves has nef canonical bundle.

Abundance Conjecture. If X is projective with K_X nef, there is a surjective holomorphic map $\Phi : X \to Y$, $\dim_{\mathbb{C}} Y = \kappa_X$, with connected fibers. The smooth fibers are Ricci-flat Kähler.

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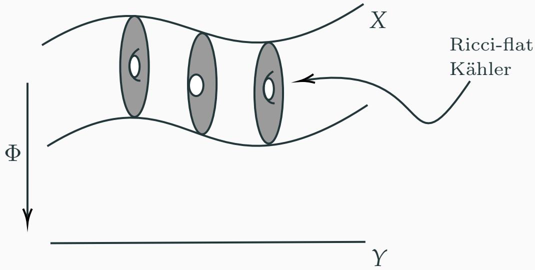

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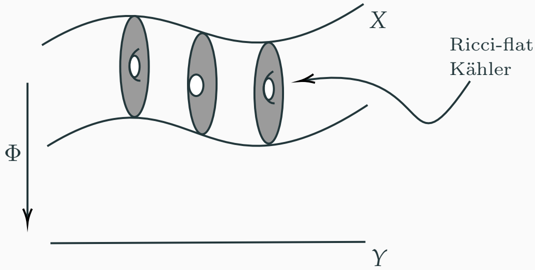

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Compact Ricci-flat Kähler manifolds

Ruling out the existence of hyperbolic Ricci-flat Kähler manifolds implies the Kobayashi–Lang conjecture for projective manifolds.

Theorem. (Beauville, Bogomolov). The universal cover of a compact Ricci-flat Kähler manifold splits as a product

$$\mathbf{C}^k \times \text{Calabi–Yau manifolds} \times \text{hyper-Kähler manifolds}.$$

Conjecture. Calabi–Yau and hyper-Kähler manifolds have rational curves $\mathbf{P}^1 \to X$.

Known for K3 surfaces, general quintic hypersurfaces in $\mathbf{P}^4$, Calabi–Yau threefolds with $b_2 > 13$, hyper-Kähler manifolds admitting a Lagrangian fibration. Hyper-Kähler manifolds with $b_2 > 3$ have an entire curve.

Theorem. The Kobayashi–Lang conjecture for projective manifolds follows from the abundance conjecture and the existence of rational curves on Calabi–Yau and hyper-Kähler manifolds.

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Lang's (Controversial) Conjecture

Obstructions to hyperbolicity are provided by rational curves $\mathbf{P}^1 \to X$ and images of abelian varieties $\mathbf{T}^n \to X$.

Conjecture. (Lang). A projective manifold with no rational curves or images of abelian varieties is hyperbolic.

The main result of the present talk shows that this controversial conjecture follows from well-established conjectures.

Theorem. (B.–Campana 2024). Lang's conjecture follows from the abundance conjecture and the existence of rational curves on (terminal) Calabi–Yau and hyperkähler varieties.

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Theorem. A projective manifold with no rational curves or images of abelian varieties is hyperbolic assuming abundance and the existence of rational curves on (terminal) CY and HK varieties.

Let $f : \mathbf{C} \to X$ be an entire curve with Zariski closure $V := \overline{f(\mathbf{C})}^{\mathrm{Zar}}$. Choose f such that $\dim_{\mathbf{C}} V$ is minimal. Replace V with its normalization V^{norm} so that singularities occur in codimension ≥ 2.

Run the MMP on V^{norm} to obtain a terminal model V^{term} which admits a holomorphic fiber space $\Phi : V^{\mathrm{term}} \to W$. Since X has no rational curves, V^{term} and the fibers of Φ are also free of rational curves. Abundance and the existence of rational curves on CY and HK varieties implies that V^{term} is an (étale quotient of an) abelian variety. The desired contradiction is obtained from showing that $V^{\mathrm{term}} \simeq V^{\mathrm{norm}}$.

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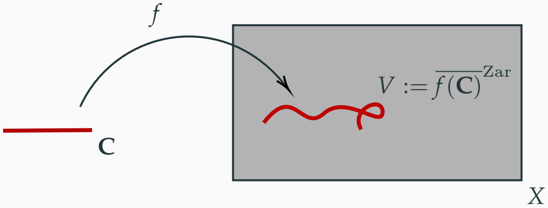

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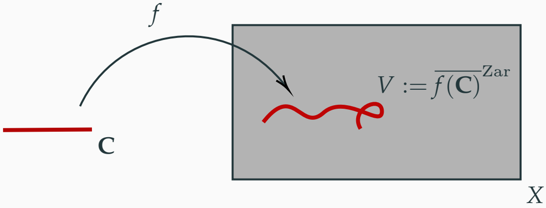

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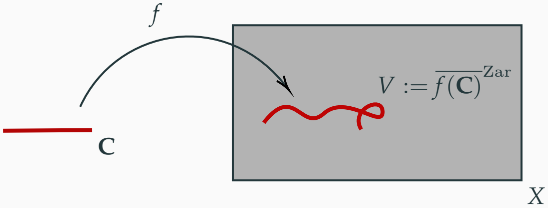

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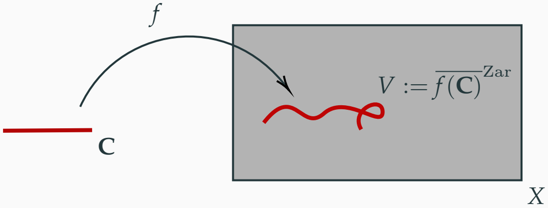

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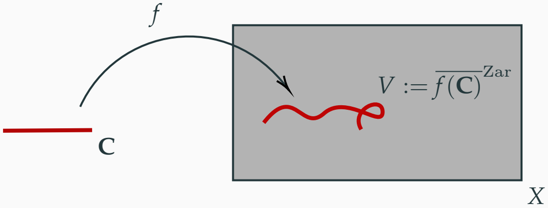

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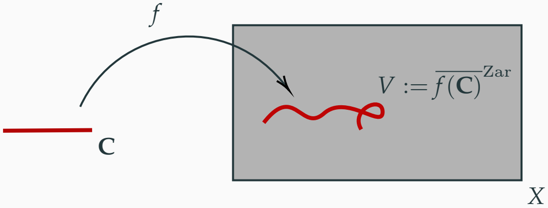

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Examples of Kähler–Einstein Non-Hyperbolic Manifolds

The Lang conjecture predicts that a projective manifold is hyperbolic if there are no rational curves $\mathbf{P}^1 \to X$ and no images from an abelian variety $\mathbf{T}^n \to X$.

We have seen that Fermat hypersurfaces are Kähler–Einstein manifolds with rational curves $\mathbf{P}^1 \to X$.

It is natural to ask if there are Kähler–Einstein manifolds with no rational curves, but are still not hyperbolic. Examples of this type among toroidal compactifications of ball quotients were found by Sarem (2023).

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Examples of Kähler–Einstein Non-Hyperbolic Manifolds

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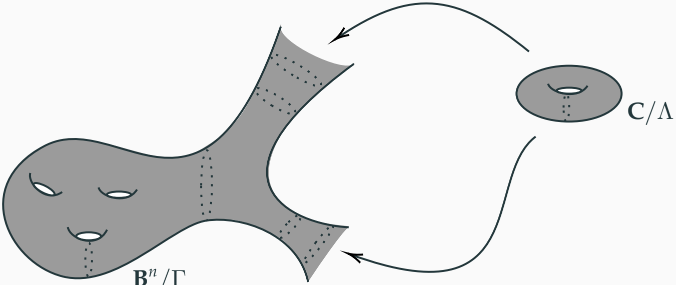

Examples of Kähler–Einstein Non-Hyperbolic Manifolds

Example	Kähler Einstein	Hyperbolic	Rational curves	Elliptic curves
Fermat hypersurfaces	Y	N	Y	Y
Sarem's examples	Y	N	N	Y
???	Y	N	N	N

With Frédéric Campana, we constructed the first example of this kind:

Theorem. (B.–Campana, 2024). There are Kähler–Einstein non-hyperbolic manifolds in any dimension with no rational or elliptic curves. In fact, the examples have Zariski dense entire curves.

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