Exponentials and Logarithms - Practice Exam 1 Solutions

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Question 1. Solve the following equation for $x \in \mathbb{R}$,

$$\left|4 - e^{x+\frac{3}{4}}\right| = 1.$$

Proof. The given equations is actually two equations:

$$4 - e^{x+\frac{3}{4}} = 1,$$

and

$$4 - e^{x+\frac{3}{4}} = -1.$$

Let us begin with the first equation. Indeed, we see that

$$\begin{aligned} 4 - e^{x+\frac{3}{4}} = 1 &\implies -e^{x+\frac{3}{4}} = -3 \\ &\implies e^{x+\frac{3}{4}} = 3 \\ &\implies x + \frac{3}{4} = \log_e(3) \\ &\implies x = \log_e(3) - \frac{3}{4}. \end{aligned}$$

Solving the second equation, we see that

$$\begin{aligned} 4 - e^{x+\frac{3}{4}} = -1 &\implies -e^{x+\frac{3}{4}} = -5 \\ &\implies e^{x+\frac{3}{4}} = 5 \\ &\implies x + \frac{3}{4} = \log_e(5) \\ &\implies x = \log_e(5) - \frac{3}{4}. \end{aligned}$$

Therefore the solutions to the given equation are $x = \log_e(3) - \frac{3}{4}$ and $x = \log_e(5) - \frac{3}{4}$. □

Question 2. Determine the value of $k \in \mathbb{R}$ such that

$$4e^x - ke^{-x} = 5,$$

has a unique solution.

Proof. Begin by letting $u = e^x$. Then

$$4e^x - ke^{-x} = 5 \implies 4u - \frac{k}{u} = 5 \implies 4u^2 - k = 5u \implies 4u^2 - 5u - k = 0.$$

It is clear that we now have a quadratic equation. Recall that a quadratic equation of this type has a unique solution (one solution) if the descriminant $\Delta = b^2 - 4ac = 0$. In our particular case,

$a = 4, b = -5$ and $c = -k$. Therefore,

$$\begin{aligned} \Delta = b^2 - 4ac = 0 \quad &\implies \quad 25 - 4(4)(-k) = 0 \\ &\implies \quad 25 + 16k = 0 \\ &\implies \quad k = -\frac{25}{16}. \end{aligned}$$

□

Question 3. Solve the following equation for $x \in \mathbb{R}$,

$$\log_x(3) + \log_3(x) = 2.$$

Proof. Begin by writing all logarithms with the same base. That is

$$\log_x(3) + \log_3(x) = 2 \quad \implies \quad \frac{1}{\log_3(x)} + \log_3(x) = 2.$$

Now let $u = \log_3(x)$. Then

$$\begin{aligned} \frac{1}{\log_3(x)} + \log_3(x) = 2 \quad &\implies \quad \frac{1}{u} + u = 2 \\ &\implies \quad 1 + u^2 = 2u \\ &\implies \quad u^2 - 2u + 1 = 0 \\ &\implies \quad (u-1)^2 = 0 \\ &\implies \quad u - 1 = 0 \\ &\implies \quad u = 1 \\ &\implies \quad \log_3(x) = 1 \\ &\implies \quad x = 3. \end{aligned}$$

□

Question 4. Graph the function $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ defined by

$$f(x) := \frac{1}{2}\log_e|x-3| + \frac{4}{5}.$$

Proof. The x-intercepts occur when $f(x) = 0$. Therefore, we see that

$$\begin{aligned} \frac{1}{2}\log_e|x-3| + \frac{4}{5} = 0 \quad &\implies \quad \frac{1}{2}\log_e|x-3| = -\frac{4}{5} \\ &\implies \quad \log_e|x-3| = -\frac{8}{5} \\ &\implies \quad |x-3| = e^{-\frac{8}{5}} \\ &\implies \quad x - 3 = \pm e^{-\frac{8}{5}} \\ &\implies \quad x = 3 \pm e^{-\frac{8}{5}}. \end{aligned}$$

The y-intercept occurs when $x = 0$. Therefore, we see that

$$y = \frac{1}{2}\log_e|-3| + \frac{4}{5} \quad = \quad \frac{1}{2}\log_e(3) + \frac{4}{5}.$$

Use an online graphing calculator to have a look at the shape of the curve. $\square$