The Schwarz Lemma in Kähler and Non-Kähler Geometry

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Hyperbolic Complex Manifolds

Definition. A complex manifold X is said to be (Brody) hyperbolic if every holomorphic map $\mathbf{C} \to X$ is constant.

Examples. The ball $\mathbf{B}^n$; the polydisk $\mathbf{D}^n$; complex manifolds with universal cover a bounded domain in $\mathbf{C}^n$; a generic smooth hypersurface in $\mathbf{P}^n$ of suitably large degree (e.g., degree $d \geq 18$ in $\mathbf{P}^3$).

Kobayashi Conjecture

One of the main questions has been the following folklore generalization of a conjecture made by Kobayashi (1973):

Conjecture. A compact hyperbolic manifold is projective and canonically polarized (i.e., the canonical bundle K_X is ample).

Evidence. Curves; Kähler surfaces (Wong '81, Campana '91); non-Kähler surfaces (assuming GSS conjecture); Compact manifolds whose universal cover is a bounded domain in $\mathbf{C}^n$; Generic smooth hypersurfaces in $\mathbf{P}^n$ of suitably large degree; manifolds of general type (i.e., K_X big).

The most significant progress over the past decade has come from differential geometry.

The Holomorphic Sectional Curvature

Hyperbolic manifolds X are characterized by the non-existence of entire curves (i.e., every holomorphic map $\mathbf{C} \to X$ is constant).

Definition. Let (X, ω) be a Hermitian manifold. The Holomorphic Sectional Curvature is defined by

$$\mathrm{HSC}_{\omega}(\xi) \ := \ \frac{1}{|\xi|_{\omega}^{4}} R(\xi, \overline{\xi}, \xi, \overline{\xi}),$$

where $\xi \in T^{1,0}X$.

Theorem. (Grauert–Reckziegel, '65). A Hermitian manifold (X, ω) with $\mathrm{HSC}_{\omega} \leq -\kappa_0 < 0$ is (Kobayashi) hyperbolic.

The condition $\mathrm{HSC}_\omega < 0$ does not characterize compact hyperbolic manifolds. Examples of projective hyperbolic surfaces with no Hermitian metric of $\mathrm{HSC}_\omega < 0$ were constructed by Demailly ('97).

It is unknown, however, how many Kobayashi hyperbolic manifolds have metrics with $\mathrm{HSC}_\omega < 0$, even for surfaces.

Theorem. (Cheung, B.–). A compact Kähler–Einstein surface $(X, \omega_{\mathrm{KE}})$ with $\mathrm{HSC}_{\omega_{\mathrm{KE}}} < 0$ satisfies $c_2 \leq 3c_1^2$. In particular, Barlow, Burniat, Campadelli, Catanese, Godeaux, Horikawa, Keum–Naie, Oliverio, Todorov surfaces do not have KE metrics with $\mathrm{HSC}_{\omega_{\mathrm{KE}}} < 0$.

Positive Holomorphic Sectional Curvature

We have seen that

$$\mathrm{HSC}_\omega \leq -\kappa_0 < 0 \implies \text{Hyperbolic.}$$

Theorem. (X. Yang, 2018). A compact Kähler manifold (X, ω) with $\mathrm{HSC}_\omega > 0$ is projective and rationally connected, i.e., any two points are contained in the image of a rational curve $\mathbf{P}^1 \to X$.

Example. The Hopf surface $\mathbf{S}^1 \times \mathbf{S}^3$ has $\mathrm{HSC}_\omega > 0$ but no rational curves at all!

The Wu–Yau Theorem

The following breakthrough on the Kobayashi conjecture was made by Heier–Lu–Wong (2010), Wu–Yau (2016), Tossatti–Yang (2017):

Theorem. Let (X, ω) be a compact Kähler manifold with $\mathrm{HSC}_{\omega} < 0$. Then X is projective and canonically polarized (K_X is ample).

In particular,

$$\mathrm{HSC}_{\omega} < 0 \implies \exists \omega_{\mathrm{KE}} \text{ such that } \mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}.$$

The Kähler–Einstein metric in the Wu–Yau theorem is constructed either from a complex Monge–Ampère equation or as the long-time solution of the Kähler–Ricci flow.

The crux of the argument is to obtain a uniform second-order estimate

$$C^{-1}\omega_h \ \leq \ \omega_t \ \leq \ C\omega_h.$$

Since $\mathrm{tr}_{\omega_t}(f^*\omega_h) = |\partial f|^2$, where $f : (X, \omega_t) \to (X, \omega_h)$ is the identity map, the uniform estimate $C^{-1}\omega_h \leq \omega_t$ follows from an estimate on $|\partial f|^2$ (the other estimate $\omega_t \leq C\omega_h$ is gotten from the equation).

For a general holomorphic map $f : (X, \omega_X) \to (Y, \omega_Y)$, we have[1]

$$\Delta_{\omega_X} |\partial f|^2 = |\nabla \partial f|^2 + \mathrm{Ric}_{\omega_X}(\partial f, \overline{\partial f}) - R_{\omega_Y}(\partial f, \overline{\partial f}, \partial f, \overline{\partial f}).$$

If $\mathrm{Ric}_{\omega_X} \geq -C_1 \omega_X + C_2 f^* \omega_Y$ and $R_{\omega_Y}(\partial f, \overline{\partial f}, \partial f, \overline{\partial f}) \leq -\kappa_0 |\partial f|^4$, then

$$\Delta_{\omega_X} |\partial f|^2 \geq |\nabla \partial f|^2 - C_1 |\partial f|^2 + \frac{1}{r} (C_2 + \kappa_0) |\partial f|^4.$$

If X is compact, then $|\partial f|^2$ attains a maximum somewhere, and at this point, $0 \geq \Delta_{\omega_X} |\partial f|^2 \geq -C_1 |\partial f|^2 + \frac{1}{r}(C_2 + \kappa_0)|\partial f|^4$. Hence,

$$|\partial f|^2 \leq \frac{C_1 r}{C_2 + \kappa_0}.$$

[1] As stated, the formula is not literally correct. The correct formula in a local frame is

$$\Delta_{\omega_g} |\partial f|^2 = |\nabla \partial f|^2 + \underbrace{g^{i\bar{j}} R^g_{i\bar{j}k\bar{\ell}}}_{\text{Ricci}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta}.$$

Problems in the Schwarz Lemma

We saw from

$$\Delta_{\omega_X} |\partial f|^2 = |\nabla \partial f|^2 + \mathrm{Ric}_{\omega_X}(\partial f, \overline{\partial f}) - R_{\omega_Y}(\partial f, \overline{\partial f}, \partial f, \overline{\partial f}),$$

that we require a lower bound on Ric_{ω_X} and an upper bound on $R_{\omega_Y}(\partial f, \overline{\partial f}, \partial f, \overline{\partial f})$.

For holomorphic maps of rank $r > 1$, the target curvature term $R_{\omega_Y}(\partial f, \overline{\partial f}, \partial f, \overline{\partial f})$ is not the Holomorphic Sectional Curvature.

Royden showed that the target curvature term $R_{\omega_Y}(\partial f, \overline{\partial f}, \partial f, \overline{\partial f})$ can be controlled by the Holomorphic Sectional Curvature if the metric is Kähler:

Theorem. (Royden '80). Let $f : (X, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map between Kähler manifolds. Suppose $\mathrm{Ric}_{\omega_g} \geq -C_1\omega_g + C_2 f^*\omega_h$ and $\mathrm{HSC}_{\omega_h} \leq -\kappa_0$. Then

$$\Delta_{\omega_g} |\partial f|^2 \ \geq \ -C_1 |\partial f|^2 + \frac{1}{r}(\kappa_0 + C_2)|\partial f|^4,$$

where $r = \mathrm{rank}(\partial f)$. In particular, if X is compact, then

$$\mathrm{tr}_{\omega_g}(f^*\omega_h) \ = \ |\partial f|^2 \leq \frac{C_1 r}{(\kappa_0 + C_2)}.$$

Royden's Schwarz lemma is the backbone of the Wu–Yau theorem (2015):

<u>Theorem</u>. Let (X, ω) be a compact Kähler manifold with a Kähler metric with $\mathrm{HSC}_\omega < 0$. Then X is projective and canonically polarized (K_X is ample).

Non-Kähler Hermitian Metrics

It is natural to consider non-Kähler Hermitian metrics, even on Kähler manifolds.

Examples. The Killing metric on the projective flag manifold $F_{1,2,3}(\mathbf{C}^3) := \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1)^3)$ is Hermitian, but not Kähler (K. Yang, '94); the Page metric ('79) on $\mathbf{P}^2 \sharp \overline{\mathbf{P}^2}$ and Chen–LeBrun–Weber metric (2008) on $\mathbf{P}^2 \sharp 2\overline{\mathbf{P}^2}$ are Hermitian, Einstein, conformally Kähler, but are not Kähler.

For a long time it was falsely believed that the target curvature term was controlled from an upper bound on the HSC, even for Hermitian non-Kähler metrics.

This was properly corrected by X. Yang and F. Zheng (2017), where they introduced:

Definition. Let (X, ω) be a Hermitian manifold. The Real Bisectional Curvature is defined

$$\text{RBC}_{\omega}(\xi) \ := \ \frac{1}{|\xi|_{\omega}^2} \sum R_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^{\alpha\overline{\beta}} \xi^{\gamma\overline{\delta}},$$

for ξ a Hermitian $(1,1)$–tensor

For Kähler metrics, the RBC is comparable to the HSC (they always have the same sign). The RBC is stronger, in general, but does not control the Ricci curvatures.

The Yang–Zheng Schwarz Lemma for Hermitian Metrics

Theorem. (Yang–Zheng, 2017). Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map of rank r between Hermitian manifolds. Suppose $\text{Ric}_{\omega_g}^{(2)} \geq -C_1 \omega_g + C_2 f^* \omega_h$ and $\text{RBC}_{\omega_h} \leq -\kappa_0 \leq 0$. Then if X is compact,

$$|\partial f|^2 \leq \frac{C_1 r}{C_2 + \kappa_0}.$$

Corollary. Let X be a compact Kähler manifold with a Hermitian metric with $\text{RBC}_{\omega_h} < 0$. Then X is projective and canonically polarized.

More General Hermitian Connections

All the results concerning the Schwarz lemma have been for the Chern connection ${}^c\nabla$ – the unique Hermitian connection compatible with the holomorphic structure $\nabla^{0,1} = \bar{\partial}$.

It is therefore natural to consider the Schwarz lemma for more general Hermitian connections.

Definition. The Gauduchon connections ${}^t\nabla$ (for $t \in \mathbf{R}$) are defined by

$$ {}^t\nabla \ := \ t\,{}^c\nabla + (1-t)\,{}^\ell\nabla, $$

where ${}^\ell\nabla$ is the Lichnerowicz connection (the restriction of the complexified Levi-Civita connection to $T^{1,0}X$).

† The Bismut connection ${}^b\nabla = {}^{-1}\nabla$ is the unique Hermitian connection with totally skew-symmetric torsion.

† The Hermitian conformal connection ${}^{\mathrm{Hc.}}\nabla = {}^{\frac{1}{2}}\nabla$ is the unique Hermitian connection whose torsion satisfies the Bianchi identity.

† The minimal connection ${}^{\min}\nabla = {}^{\frac{1}{3}}\nabla$ is the unique Hermitian connection that achieves the minimum of the map $\nabla \mapsto |{}^\nabla T|^2$

A Monotonicity Theorem

Theorem. (B.–Stanfield). Let (X, ω) be a Hermitian manifold. Then the Gauduchon Holomorphic Sectional Curvature satisfies

$$ {}^t\mathrm{HSC}_\omega \leq {}^c\mathrm{HSC}_\omega - \frac{(1-t)^2}{4} |{}^cT|^2, $$

where ${}^cT$ denotes the Chern torsion.

In particular, ${}^c\mathrm{HSC}_\omega < 0$ is the strongest curvature constraint, while ${}^c\mathrm{HSC}_\omega > 0$ is the weakest. This offers an explanation for the significant difference in their geometric consequences:

- † ${}^c\mathrm{HSC}_\omega \leq -\kappa_0 < 0 \implies$ Hyperbolic (even if ω is not complete).
- † $\mathrm{HSC}_\omega > 0 \implies$ Rationally connected (for compact Kähler); for non-Kähler metrics, $\mathrm{HSC}_\omega > 0$ does not imply the existence of any rational curves.

Guiding Principle

Instead of computing directly in coordinates, one should work with a more general Bochner formula. We want to work abstractly for as long as possible before descending into the wilderness of local coordinates.

For the Chern connection, this manifests as

$$\Delta_\omega |\sigma|^2 \ = \ |\nabla \sigma|^2 - \{\Theta^{(\mathcal{E},h)}(\sigma), \overline{\sigma}\},$$

where $\sigma \in H^0(\mathcal{E})$ is a holomorphic section and $\Theta^{(\mathcal{E},h)}$ is the curvature of h.

For the Schwarz lemma, $\sigma = \partial f$ and $\mathcal{E} = \Omega_X^{1,0} \otimes f^* T^{1,0} Y$.

A General Bochner Formula

Theorem. (B.–Stanfield). Let $(\mathcal{E}, h) \longrightarrow X$ be a holomorphic vector bundle over a Hermitian manifold (X, ω). Let ∇ be a Hermitian connection on $\mathcal{E}$. Then for any holomorphic section $\sigma \in H^0(\mathcal{E})$, we have

$$\Delta_\omega |\sigma|_h^2 \;=\; |\nabla^{1,0}\sigma|^2 + |\nabla^{0,1}\sigma|^2 + 2\mathrm{Re}\{\nabla^{1,0}\nabla^{0,1}\sigma, \overline{\sigma}\} - \{\Theta^{(\mathcal{E},h)}\sigma, \overline{\sigma}\}.$$

Theorem. (B.–Stanfield). Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Endow that source manifold with ${}^s\nabla$ and the target manifold with ${}^t\nabla$, where $s, t \in \mathbf{R} \backslash \{0, 1/2\}$. Then

$$\begin{aligned}
\Delta_{\omega_g} |\partial f|^2 \quad \geq \quad & \frac{(s-1)^2}{2s(2s-1)} {}^s\mathrm{Ric}_g^{(1)} + \frac{s^2+2s-1}{2s(2s-1)} {}^s\mathrm{Ric}_g^{(2)} \\
& + \frac{(s-1)}{2(2s-1)} \left({}^s\mathrm{Ric}_g^{(3)} + {}^s\mathrm{Ric}_g^{(4)} \right) + \frac{(s-1)^3}{4(2s-1)} \mathrm{Re} \left(T_g \otimes \overline{T_g} \right) \\
& + \frac{(s-1)(1-s-s^2-3s^3)}{8s(2s-1)} |T_g|^2 + \frac{(s-1)^2(1-2s-3s^2)}{8s(2s-1)} |T_g|^2 \\
& - \frac{t}{2t-1} {}^t\mathrm{RBC}_{\omega_h} - \frac{(t-1)}{(2t-1)} {}^t\widetilde{\mathrm{RBC}}_{\omega_h} + \frac{(t-1)^2}{4(2t-1)} |T_h|^2 \\
& - \frac{(t-1)^2(t^2+2t-1)}{8t(2t-1)} |T_h|^2 - \frac{(t-1)^4}{8t(2t-1)} |T_h|^2 \\
& + \underbrace{(2st-s-t)}_{\text{Zhao–Zheng duality}} \mathrm{Re} \left(T_g \otimes \overline{T_h} \right).
\end{aligned}$$

The Bismut Schwarz Lemma

Endow the source and target with the Bismut connection ${}^{b}\nabla = {}^{-1}\nabla$:

$$\begin{aligned} \Delta_{\omega_g} |\partial f|^2 \quad \geq \quad & \frac{2}{3} {}^{b}\mathrm{Ric}_g^{(1)} - \frac{1}{3} {}^{b}\mathrm{Ric}_g^{(2)} + \frac{1}{3} \left({}^{b}\mathrm{Ric}_g^{(3)} + {}^{b}\mathrm{Ric}_g^{(4)} \right) \\ & + \frac{2}{3} \mathrm{Re} \left(T_g \otimes \overline{T_g} \right) - \frac{1}{3} |T_g|^2 \\ & - \frac{2}{3} {}^{b}\mathrm{RBC}_{\omega_h} - \frac{1}{3} {}^{b}\widetilde{\mathrm{RBC}}_{\omega_h} + \frac{1}{3} |T_h|^2 \\ & + \frac{1}{3} |T_h|^2 - \frac{2}{3} |T_h|^2 + 4\mathrm{Re} \left(T_g \otimes \overline{T_h} \right). \end{aligned}$$

The Strominger–Bismut Schwarz Lemma

Theorem. (B.–Stanfield). Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a rank r holomorphic map between Hermitian manifolds. Suppose that

$$2\,{}^b\mathrm{Ric}^{(1)}_{\omega_g} - {}^b\mathrm{Ric}^{(2)}_{\omega_g} + {}^b\mathrm{Ric}^{(3)}_{\omega_g} + {}^b\mathrm{Ric}^{(4)}_{\omega_g} \;\geq\; -C_1\omega_g + C_2 f^*\omega_h,$$

the Chern torsions are bounded by $|T_g|^2 \leq \Lambda_0$ and $|T_h|^2 \leq \Lambda_1$, and the Bismut Real Bisectional Curvatures are bounded by

$${}^b\mathrm{RBC}_{\omega_h} \;\leq\; -\kappa_0, \qquad {}^b\widetilde{\mathrm{RBC}}_{\omega_h} \;\leq\; -\kappa_1.$$

If $C_2 r - \Lambda_1 + \kappa_0 + 2\kappa_1 > 0$, then

$$|\partial f|^2 \;\leq\; \frac{r(C_1 + \Lambda_0)}{C_2 r - \Lambda_1 + \kappa_0 + 2\kappa_1}.$$

An Improvement on the Schwarz Lemma

The anti-symmetric component of $\nabla^{1,0}\partial f$ yields a torsion term for both the source and target metric.

This can be used to lessen the strain on the curvature terms for the source and target metrics, using the Peter–Paul inequality:

$$|{}^c\nabla\partial f|^2 \quad \geq \quad \frac{1}{4}(1-\tau^{-1})|T_g|^2 + \frac{1}{4}(1-\tau)|T_h|^2,$$

where $\tau \in [0, +\infty]$.

Definition. For a constant $\tau > 0$, we define the Tempered Real Bisectional Curvature

$$^{c}\mathrm{RBC}_{\omega}^{\tau} \ := \ ^{c}\mathrm{RBC}_{\omega}^{\tau} - \frac{1}{4}(1-\tau)\,\mathcal{Q}_{\omega},$$

where $\mathcal{Q}_{\omega}$ is a positive-definite term that is quadratic in the (Chern) torsion.

In a local frame, $\mathcal{Q}_{i\bar{i}j\bar{j}k\bar{\ell}} = T_{ik}^{p}\overline{T_{j\ell}^{q}}g_{p\bar{q}}$.

Theorem. (B.–Stanfield). Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map of rank r from a Kähler manifold to a Hermitian manifold. If ${}^c\mathrm{Ric}_{\omega_g} \geq -C_1\omega_g + C_2 f^*\omega_h$ and ${}^c\mathrm{RBC}^{\tau}_{\omega_h} \leq -\kappa_0 \leq 0$, then

$$\Delta_{\omega_g}|\partial f|^2 \ \geq \ -C_1|\partial f|^2 + \frac{1}{r}(C_2 + \kappa_0)|\partial f|^4.$$

Hence, if X is compact, and $C_2 + \kappa_0 > 0$, we have

$$|\partial f|^2 \quad \leq \quad \frac{C_1 r}{C_2 + \kappa_0}.$$

Theorem. (B.–Stanfield). Let X be a compact Kähler manifold with a Hermitian metric satisfying ${}^c\mathrm{RBC}^{\tau}_{\omega} < 0$. Then X is projective and canonically polarized.

In particular, X admits a Kähler–Einstein metric with negative Ricci curvature.

The Tempered Ricci Curvature

If the source metric is not Kähler, we have the following tempered version of the Second Chern Ricci Curvature:

Definition. For a constant $\tau > 0$, we define the Tempered Ricci Curvature

$$ {}^{c}\mathrm{Ric}_{\omega}^{\tau} \ := \ {}^{c}\mathrm{Ric}_{\omega}^{(2)} + \frac{1}{4}\left(1 - \frac{1}{\tau}\right)\mathcal{Q}_{\omega}^{2}, $$

where $\mathcal{Q}_{\omega}^{2}$ is a positive-definite term that is quadratic in the (Chern) torsion.

In a local frame, ${}^{c}\mathrm{Ric}_{k\bar{\ell}}^{(2)} := g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}$ and $\mathcal{Q}_{k\bar{\ell}}^{2} = g^{i\bar{j}} g^{p\bar{q}} T_{ip\bar{\ell}} \overline{T_{jq\bar{k}}}$.

The Tempered Hermitian Curvature Flow

The Tempered Ricci curvature motivates the study of the following 'Tempered Hermitian Curvature Flow':

$$\frac{\partial \omega_t}{\partial t} = -{}^c\mathrm{Ric}^{(2)}_{\omega_t} - \frac{1}{4}(1-\tau^{-1})\mathcal{Q}^2_{\omega_t} - \omega_t.$$

This is very close to the Hermitian Curvature Flow that was studied by Ustinovskiy (2018) and Fei–Phong (2019).

Question. Let (X, ω) be a compact Hermitian manifold with ${}^c\mathrm{RBC}^{\tau}_{\omega} < 0$. Does the Tempered Hermitian Curvature Flow exist for all time? Does it converge to a Kähler current?