NON-STANDARD EINSTEIN METRICS ON PROJECTIVE SPACE

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During Jan Nienhaus' lecture[1] as part of the *Virtual Seminar on Geometry with Symmetries*, Ramiro Lafuente asked the following:

Question. Are there non-standard Einstein metrics on complex projective space $\mathbf{P}^n$?

The standard Fubini–Study metric on $\mathbf{P}^n$ is homogeneous with respect to the transitive $\mathrm{U}(n+1)$–action on $\mathbf{P}^n$. By a theorem of Matsushima [10], any Kähler–Einstein metric is biholomorphically isometric to the Fubini–Study metric. Further, by a theorem of Hirzebruch–Kodaira [4] and Yau [15], any Kähler manifold homeomorphic to $\mathbf{P}^n$ is biholomorphic to $\mathbf{P}^n$ (see [13] for a lovely exposition). So any non-standard Einstein metric is necessarily non-Kähler.

LeBrun [6] has shown that any Einstein Riemannian metric that is Hermitian in the sense that

$$g(\cdot, \cdot) = g(J\cdot, J\cdot)$$

must be conformally Kähler, i.e., $g = f^2 h$ for some Kähler metric h and smooth positive function f. The argument is primarily local with compactness only being used to rule out the possibility of the metric being anti-self-dual. In particular, it follows that any anti-self-dual Einstein metric on $\mathbf{P}^2$ is the Fubini–Study metric. A classification of the compact anti-self-dual Kähler surfaces was given by Itoh [5, Theorem 4].

LeBrun [7] later used this to classify all Einstein metrics on compact complex surfaces that are Hermitian with respect to some integrable complex structure. They are Kähler–Einstein with exactly two exceptions: The Page metric [12] on $\mathbf{P}^2 \sharp \overline{\mathbf{P}^2}$ (cohomogeneity one) and the Chen–LeBrun–Weber metric [2] on $\mathbf{P}^2 \sharp 2\overline{\mathbf{P}^2}$ (toric, i.e., cohomogeneity two).

In the absence of any Hermitian condition, the uniqueness of Einstein metrics on $\mathbf{P}^2$ is much less understood. Ziller [16] showed that there are no homogeneous Einstein metrics on $\mathbf{P}^{2n}$ for $n \geq 1$. Moreover, Patrick Donovan, an undergraduate student under the supervision of Timothy Buttsworth at The University of Queensland, has investigated cohomogeneity-one

[1]Here: https://www.youtube.com/watch?v=ByKpoU8MuII&t=0s

Einstein metrics on compact simply connected 4–manifolds. The analysis involves numerically solving a system of differential equations with certain boundary conditions. The numerical calculations appear to indicate that there are no cohomogeneity-one Einstein metrics on $\mathbf{P}^2$.

Gursky–LeBrun [3] showed that the only Einstein metric of positive sectional curvature on $\mathbf{P}^2$ is the Fubini–Study metric. LeBrun [8] showed that the Fubini–Study metric is the unique Einstein metric on $\mathbf{P}^2$ with $\mathcal{W}_+(\omega, \omega) > 0$, where $\mathcal{W}_+$ is the self-dual-part of the Weyl curvature, and ω is the (essentially) unique harmonic 2–form with respect to the given metric. Wu [14] showed that the Fubini–Study metric is the unique Einstein metric on $\mathbf{P}^2$ that satisfies $\det(\mathcal{W}_+) > 0$, where $\mathcal{W}_+$ is identified as an endomorphism of the self-dual 2–forms Λ^+. A more transparent proof of Wu's result was subsequently given by LeBrun [9].

Question. How many connected components does the moduli space of Einstein metrics on $\mathbf{P}^2$ have?

Anderson [1, Theorem D] has shown that each component of the moduli space of Einstein metrics with positive Ricci curvature on $\mathbf{P}^2$ is compact in the $\mathcal{C}^\infty$–topology. Further, there are only finitely many components of the space of Einstein metrics with $\text{Ric}(g) = g$ and $\text{vol}(\mathbf{P}^2, g) \geq c > 0$.

Question. Are there Einstein metrics of cohomogeneity two on $\mathbf{P}^2$? Are there almost Hermitian Einstein metrics on $\mathbf{P}^2$ that are not Kähler–Einstein?

Higher Dimensions. The groups acting transitively on $\mathbf{P}^n$ were classified by Oniscik [11, p. 168], and we see that $\mathbf{P}^n = \text{SU}(n+1)/\text{S}(\text{U}(1) \times \text{U}(n))$, in which case the isotropy representation is irreducible, while $\mathbf{P}^{2n+1} = \text{Sp}(n+1)/(\text{Sp}(n) \times \text{U}(1))$, in which case, the isotropy representation is $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. In the first case, the only invariant metric is the standard Fubini–Study metric. In the second case, $\dim \mathfrak{p}_1 = 2$, $\dim \mathfrak{p}_2 = 4n$, and $\text{U}(1)$ acts by a rotation on $\mathfrak{p}_1$ and trivially on $\mathfrak{p}_2$ and $\text{Sp}(n)$ acts trivially on $\mathfrak{p}_1$ and by its standard representation on $\mathfrak{p}_2 \simeq \mathbf{H}^n$. Hence, there is a two-parameter family of homogeneous metrics on $\mathbf{P}^{2n+1}$. Since the complex structure leaves the splitting of $\mathfrak{p}$ invariant, these metrics are Hermitian.

On odd-dimensional complex projective spaces $\mathbf{P}^{2n+1}$, Ziller [16, p. 356–357] showed that there is a non-standard, homogeneous, Hermitian Einstein metric with positive sectional curvature. Ziller also determines the pinching of the sectional curvature to be $\delta = 1/4(n+1)^2$.

Question. Are there non-standard Einstein metrics on even-dimensional projective spaces $\mathbf{P}^{2n}$?

Question. Does there exist an Einstein metric of negative Ricci curvature on $\mathbf{P}^n$?

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